NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3598

METHOD AND TABLES FOR DETERMINING THE TIME RESPONSE TO A UNIT IMPULSE FROM FREQUENCY-RESPONSE DATA AND FOR DETERMINING THE FOURIER TRANSFORM OF A FUNCTION OF TIME

By Carl R. Huss and James J. Donegan

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
January 1956

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3598

METHOD AND TABLES FOR DETERMINING THE TIME RESPONSE TO A UNIT IMPULSE FROM FREQUENCY-RESPONSE DATA AND FOR DETERMINING THE FOURIER TRANSFORM OF A FUNCTION OF TIME

By Carl R. Huss and James J. Donegan

SUMMARY

A simple and rapid method is presented for the determination of the time response to a unit impulse from frequency-response data and for evaluating the Fourier transform of a function of time. Both methods are applicable to linear functions for which Fourier transforms exist. Tables are presented which greatly reduce the time required to perform the computations. Procedures for performing the calculations by use of the tables are illustrated and outlined step by step.

INTRODUCTION

The direct and inverse Fourier transforms are at present being used extensively in the analysis of flight test data. The direct transform provides the means of determining the frequency response of a system from a transient response to an input. The inverse transform provides the means of determining the time response to either a unit impulse or an arbitrary input from the frequency response of a system. In both cases the transfer of data between the time plane and the frequency plane is accomplished without knowledge of the transfer function relating the input and output.

Some recent experience in analyzing and extending flight measurements obtained with a swept-wing bomber has led to the establishment of a simplified method similar to that of reference 1 of determining the time response to a unit impulse from frequency-response data and of determining the Fourier transform of a function of time. The simplification of the methods results from the use of prepared tables. Since these methods and tables should be of help to others engaged in the analysis of linear systems, they are included in the present paper along with explanatory material of the mathematical principles involved, examples showing the application of the method and tables, and several comparisons illustrating the accuracy of the procedures.

SYMBOLS

A	modulus of Fourier transform
$F(i\omega)$	Fourier transform, $\int_0^\infty f(t)e^{-i\omega t}dt$
$f(t)$	function of time
$H(i\omega)$	frequency response
$h(t)$	time response to a unit impulse (total unless with subscript n)
I	imaginary component
n	interval
R	real component
r	amplitude
t or τ	time, sec
x	output
z	normalizing parameter, $\frac{\Delta\omega}{2}t$ or $\frac{\Delta t}{2}\omega$
Δ	incremental values
Φ	phase angle of Fourier transform, deg
ω	frequency, radians/sec
δ	input

Subscripts:

n	refers to particular rectangle
x	output
δ	input
max	maximum

Absolute value of a term is denoted by $| \; |$

METHOD

The methods presented in this paper are applicable to linear systems for which Fourier transforms exist. The methods are presented in the following order: (1) determination of the time response to a unit impulse from frequency-response data and (2) determination of the Fourier transform of a function of time. Also included as appendix A is a procedure for numerically evaluating Duhamel's integral with an integrating matrix based on the Newton-Cotes quadrature formulas given in reference 2 which permit the determination of the response of a system to an arbitrary input when the response to a unit impulse is known.

Determination of Time Response to a Unit Impulse From Frequency-Response Data

As indicated by George F. Floyd in reference 1, the time response to a unit impulse $h(t)$ can be related to the frequency response by

$$h(t) = \frac{2}{\pi} \int_0^\infty R\left[H(i\omega)\right] \cos \omega t \, d\omega \tag{1}$$

where $R\left[H(i\omega)\right]$ is the real part of the frequency response.

The integration of equation (1) is sometimes difficult since an analytic expression for $R\left[H(i\omega)\right]$ is often not known and, when one is known, it can be very complicated. Numerical values of $R\left[H(i\omega)\right]$, however, are easily obtained from the amplitude ratio and cosine of the phase angle of the frequency response and, therefore, the problem lends itself well to some method of numerical integration.

The numerical method employed by Floyd to solve equation (1) was to approximate the $R\left[H(i\omega)\right]$ with straight lines. This type of approximation reduces equation (1) for each straight-line segment to the simple form of $\sin x/x$ for which tables existed. The total response is then the sum of the individual response for each straight-line segment. This method gives good results for simple systems; however, when applied to a system with several high-frequency modes, it requires considerable time.

The method suggested in this paper is to approximate the $R\left[H(i\omega)\right]$ with a staircase type of function having equal frequency intervals and of

such height that the area under each step of the staircase function equals the area under that portion of the $R\left[H(i\omega)\right]$ curve within the interval.

Equation (1) can then be written as

$$h(t) = \frac{2}{\pi}\sum_{n=1}^{\infty} h_n(t) \tag{2}$$

where

$$h_n(t) = \int_{\omega_{n-1}}^{\omega_n} R\left[H(i\omega)\right]\cos \omega t \, d\omega \tag{3}$$

and n, ω_n, and ω_{n-1} are shown in figure 1(a) and, for $n=1$, $\omega_{n-1} = 0$. In equation (3), $h_n(t)$ represents the contribution to the response to the unit impulse of the area of the interval between ω_{n-1} and ω_n of the staircase function. By substituting the constant amplitude r_n for $R\left[H(i\omega)\right]$ in equation (3) and performing the integration, the individual contribution becomes

$$h_n(t) = r_n \frac{2}{t} \sin\left(\frac{\Delta\omega}{2}t\right)\cos\left[(2n - 1)\frac{\Delta\omega}{2}t\right] \tag{4}$$

where r_n and $\Delta\omega$ are defined as in figure 1(a).

Examples of the $h_n(t)$ for steps of unit amplitude and a frequency interval of 1 radian per second of various locations on the frequency scale are shown in figure 2.

Equation (4) can be normalized by letting

$$z = \frac{\Delta\omega t}{2} \tag{5}$$

Then,

$$h_n(t) = \Delta\omega r_n \frac{\sin z \cos(2n - 1)z}{z} \tag{6}$$

By substituting equation (6) into equation (2), the total response is given by

$$h(t) = \frac{2}{\pi}\Delta\omega \sum_{n=1}^{\infty} r_n \frac{\sin z \cos(2n - 1)z}{z} \tag{7}$$

where the time values corresponding to the z-values for which the calculations are made depend on the $\Delta\omega$ chosen.

By tabulating the function $\frac{\sin z \cos(2n - 1)z}{z}$ for various values of z and n, the response to the unit impulse can be easily and rapidly obtained for a chosen frequency interval $\Delta\omega$. Such a tabulation is presented as table I for values of z from 0 to 2.5 and values of n from 1 to 50.

The following procedure is suggested in computing the time response to a unit impulse from frequency-response data. The steps of the procedure are outlined and illustrated by the use of tables I and II and figure 3.

(1) Choose the frequency interval to be used. The choice of $\Delta\omega$ depends on such factors as the accuracy desired, the shape, and frequency range of the $R\left[H(i\omega)\right]$. In general, best results are obtained if the interval is chosen so that the staircase function adequately represents the $R\left[H(i\omega)\right]$. (That is, within the intervals, the $R\left[H(i\omega)\right]$ is as constant as practicable). For the example shown in figure 3, a $\Delta\omega$ of 1 was chosen.

(2) Fit the staircase function to $R\left[H(i\omega)\right]$ by using the chosen $\Delta\omega$ as shown in figure 3. In this case 20 intervals were used. In order to obtain accurate results, $R\left[H(i\omega)\right]$ must either be zero or approach zero within the number of intervals used.

(3) Read and record the amplitude of each interval as shown in column 2 of table II.

(4) Read $h_n(t)$ for each interval at the desired values of z from table I and record as shown in column (3) of table II. The example is solved for $z = 0.5$ which, for $\Delta\omega = 1$ radian per second, corresponds to a time of 1.0 second.

(5) Multiply the amplitude and $h_n(t)$ together for each individual interval as shown in column (4) of table II.

(6) Sum the products of the amplitude and h_n for the individual intervals at the desired values of z. This total is shown as the summation of column (4) in table II.

(7) Multiply the summation by $2\,\Delta\omega/\pi$ to obtain the time response to a unit impulse. This product is shown at the bottom of table II and the result is shown in figure 3 at $t = 1.0$ second.

Examples of applications of the method are presented in figures 3 and 4 along with comparisons of the results with the exact time responses. In figure 3 the method has been applied to a heavily damped linear system with one low-frequency mode defined by the transfer function $\frac{x}{\delta} = \frac{1}{s^2 + 6s + 10}$. In figure 4 the method has been applied to the system of figure 3 with the addition of two higher frequency modes. This system was defined by the transfer function

$$\frac{x}{\delta} = \frac{1}{s^2 + 6s + 10}\,\frac{100}{s^2 + 0.4s + 100}\,\frac{225}{s^2 + 0.2s + 225}$$

Determination of the Fourier Transform of a Function of Time

As in the determination of the time response to an impulse from frequency-response data, the determination of the Fourier transform of a time function involves the fitting of a staircase function, in this case with constant time intervals, to the time function so that the area of the individual time intervals equals the area under the represented portion of the function. The real part and the imaginary part of the Fourier transform for the individual intervals are then determined and the results for the various intervals are summed to give the real part and imaginary part of the total Fourier transform of a time function. The modulus and phase angle of the Fourier transform are then easily obtained.

The Fourier transform of a time function is

$$F(i\omega) = \int_0^\infty f(t)\ e^{-i\omega t}dt \tag{8}$$

which for a staircase function can be written

$$F(i\omega) = \sum_{n=1}^{\infty} F_n(i\omega)$$

where

$$F_n(i\omega) = \int_{t_{n-1}}^{t_n} r_n e^{-i\omega t}dt \tag{9}$$

where r_n, t_n, and t_{n-1} are shown in figure 1(b) and for $n = 1$, $t_{n-1} = 0$.

The expression $F_n(i\omega)$ represents the contribution of the individual intervals to the total Fourier transform.

The real part of equation (9) is

$$R_n\left[F(i\omega)\right] = r_n \frac{2}{\omega} \sin\left(\frac{\Delta t}{2}\omega\right)\cos\left[(2n - 1)\frac{\Delta t}{2}\omega\right] \tag{10}$$

and the imaginary part is

$$I_n\left[F(i\omega)\right] = -r_n \frac{2}{\omega} \sin\left(\frac{\Delta t}{2}\omega\right)\sin\left[(2n - 1)\frac{\Delta t}{2}\omega\right] \tag{11}$$

where r_n and Δt are shown in figure 1(b).

The modulus of the Fourier transform for an individual interval is

$$A_n(\omega) = \sqrt{(R_n)^2 + (I_n)^2} \tag{12}$$

and the phase angle is

$$\Phi_n(\omega) = -\tan^{-1} \frac{I_n}{R_n} \tag{13}$$

Examples of the Fourier transform for a time interval of 0.1 second at various locations in the time plane are shown vectorially in figure 5.

The real part and imaginary part for the total Fourier transform of a time function is simply the sum of the real part and imaginary part of the individual intervals.

$$R\left[F(i\omega)\right] = \sum R_n\left[F(i\omega)\right] \tag{14}$$

and

$$I\left[F(i\omega)\right] = \sum I_n\left[F(i\omega)\right] \tag{15}$$

In a manner similar to the time response to a unit impulse the real and imaginary part (eqs. (10) and (11)) can be normalized by letting

$$z = \frac{\Delta t}{2}\omega \tag{16}$$

Then the real part (eq. (10)) becomes

$$R_n\left[F(i\omega)\right] = \Delta t r_n \frac{\sin z \cos(2n - 1)z}{z} \tag{17}$$

and the imaginary part (eq. (11)) becomes

$$I_n\left[F(i\omega)\right] = -\Delta t r_n \frac{\sin z \sin(2n - 1)z}{z} \tag{18}$$

The total real part is then

$$R\left[F(i\omega)\right] = \Delta t \sum r_n \frac{\sin z \cos(2n - 1)z}{z} \tag{19}$$

and the total imaginary part is

$$I\left[F(i\omega)\right] = -\Delta t \sum r_n \frac{\sin z \sin(2n - 1)z}{z} \tag{20}$$

The total modulus of the Fourier transform is

$$A(\omega) = \sqrt{\left\{R\left[F(i\omega)\right]\right\}^2 + \left\{I\left[F(i\omega)\right]\right\}^2} \tag{21}$$

and the total phase angle is

$$\Phi(\omega) = -\tan^{-1} \frac{I\left[F(i\omega)\right]}{R\left[F(i\omega)\right]} \tag{22}$$

Again, as in the case of the time response to a unit impulse, the calculation of the Fourier transform of a time function can be simplified by the use of tabulated values of the function $\frac{\sin z \cos(2n - 1)z}{z}$ and $\frac{\sin z \sin(2n - 1)z}{z}$ for various values of n and z. It should be noted

that the function $\frac{\sin z \cos(2n - 1)z}{z}$ is the same function that was tabulated to calculate the time response to a unit impulse. The real part of the Fourier transform can, therefore, be calculated from the same table. It is only necessary, then, that the additional function $\frac{\sin z \sin(2n - 1)z}{z}$ be tabulated. This tabulation has been done for values of z from 0 to 1.0 and values of n from 1 to 50 and is presented as table III.

In order to interpolate between the z-values of tables I and III, plots can be made for each interval by using the values given in the table.

Fourier transforms of lengthy time functions using a small time increment Δt may be computed by using the tables in conjunction with the shifting theorem. This theorem states that, if the Fourier transform $F(i\omega)$ of a time function $f(t)$ exists, the Fourier transform of $f(t - \tau)$ is $e^{-i\omega\tau}F(i\omega)$. Therefore, a time function extending beyond the limits of the table in time may be partitioned into segments, the first extending from 0 to τ_1, the second extending from τ_1 to τ_2, the third extending from τ_2 to τ_3, and so forth. The total Fourier transform of the time function is simply the vectorial sum of the Fourier transforms of each segment. The Fourier transform of the first segment is obtained directly from the tables. The Fourier transform of each succeeding segment is obtained by assuming the origin to be at τ and then multiplying the results obtained from application of the tables by $e^{-i\omega\tau}$.

The following procedure is suggested for computing the Fourier transform of a function of time. The steps of the procedure are outlined and illustrated by the use of table I, table III, table IV, and figure 6.

(1) Choose the time interval to be used. The Δt used depends on the highest frequency desired from the Fourier transform. If it is assumed that one complete oscillation in the time plane can be defined by no less than 4 points, the maximum frequency for which accurate results can be expected will be

$$\omega_{max} = \frac{2\pi}{3\,\Delta t} = \frac{2.094}{\Delta t} \tag{23}$$

When the desired maximum frequency does not dictate the interval to be used, then the interval should be chosen so that the time function is as constant as practicable within the interval. For the example shown in figure 6, a Δt of 0.1 second was chosen.

(2) Fit staircase functions to the time function as shown in figure 6. In order to obtain accurate results at the low frequencies, the function of time must damp to practically zero within the number of intervals used. For this example 25 intervals were used.

(3) Read and record amplitude of each interval as shown in column (2) of table IV.

(4) Read the values for the calculation of the real part of the Fourier transform for each interval at the desired z-values from table I and record in column (3) of table IV. The example is solved for $z = 0.25$ which for $\Delta t = 0.1$ corresponds to $\omega = 5$ radians per second.

(5) Read the values for the calculation of the imaginary part of the Fourier transform for each interval at the desired z-values from table III and record as shown in column (4) of table IV. The example is solved for $z = 0.25$ which for $\Delta t = 0.1$ corresponds to $\omega = 5$ radians per second.

(6) Multiply the amplitude and the values for the real part and the amplitude and the values for the imaginary part together for each individual interval as shown in columns (5) and (6), respectively, of table IV.

(7) Sum the products of the amplitude and the values for the real part and the amplitude and the values for the imaginary part. These totals are shown in table IV as the summation of columns (5) and (6), respectively.

(8) Use equation (19) to compute the real part of the total Fourier transform and equation (20) to compute the imaginary part of the total Fourier transform. These computations are demonstrated at the bottom of table IV.

(9) Use equation (21) to obtain the amplitude of the Fourier transform and equation (22) to obtain the phase angle of the Fourier transform. This calculation is demonstrated at the bottom of table IV and the results obtained are shown in figure 6 at $\omega = 5$ radians per second.

An example of the application of this method to a time function and a comparison of the results with the exact amplitude and phase angle is shown in figure 6. The time function shown in figure 6 is the same time response to a unit impulse as is shown in figure 3. In figure 6 the Fourier transform of this time function was obtained and in figure 3 the

real part of the Fourier transform was used to obtain the time function of the inverse Fourier transform.

DISCUSSION

As was mentioned in the "Method" section, in order for the methods of this paper to give correct results, the staircase function must adequately represent the function being fitted. This will be the case if (1) the interval used is not too large, (2) the function being fitted approaches zero within the number of intervals used, and (3) the area under the staircase function within the interval equals the area under the represented portion of the function being fitted.

When the time response to a unit impulse is obtained, too large an interval may mask some of the modes present in the frequency response; thus, in the time response to the unit impulse, such modes would not be properly represented either as to frequency or amplitude. If the real part of the frequency response does not approach zero within the number of intervals used, large errors in $h(t)$ will be obtained near zero time. This result is demonstrated in figure 3 where the error at zero time could be reduced to 0.0080 by using 50 intervals and fitting the curve to a frequency of 50 radians per second. If the area of the staircase function does not equal the area of the real part of the frequency response for each interval, then, of course, inaccuracies will be obtained throughout the calculations.

An idea of the accuracy of the computations involved in determining the time response to a unit impulse by the method and tables of this paper can be obtained from figure 3 for a simple system and from figure 4 for a system with multiple modes. The overall accuracy would, of course, be increased either by using a smaller frequency interval or by using more intervals.

The total time required to obtain response to a unit impulse such as that shown in figures 3 and 4 by application of the method presented herein and using a desk calculator is about 1.5 hours.

In the case of obtaining the Fourier transforms of a function of time, the interval chosen determines the maximum frequency that can be expected to be determined accurately from the method. The larger the interval, the lower will be the maximum frequency obtained. If the time function does not approach zero within the number of intervals used, an end correction must be added in order to obtain the correct Fourier transform. The greatest inaccuracies, however, will be in the very low frequency range near zero radians per second. As in the previous case if the area under the staircase function does not equal the area of the time function for each interval, inaccuracies are obtained throughout the calculations.

The accuracy of the calculations involved in evaluating the Fourier transform by the method and tables of this paper can be seen in figure 6. As before, the overall accuracy could, of course, be increased by using a smaller time interval but, in view of the comparison shown, it would not seem to be necessary for the function shown.

The time required if a desk calculator and the method and tables of this paper are used to evaluate the Fourier transform of a time function such as shown in figure 6 is about 2.0 hours.

It should be noted that in some cases the values of z given in the tables may not correspond to sufficient time or frequency values to define accurately the $h(t)$ or $F(i\omega)$. The values of time and frequency corresponding to the z-values in the tables, as are shown by equations (5) and (16), depend on the interval chosen. The functions $\frac{\sin z \cos(2n - 1)z}{z}$ and $\frac{\sin z \sin(2n - 1)z}{z}$ can be determined for values of z not given in the tables by interpolation (that is, by plotting the numbers given in the table for each interval and reading the value of the function of the desired z from the plots).

The time response to an impulse when used with Duhamel's integral permits the time response to any arbitrary input to be computed; hence, the response of a given system to inputs beyond those normally or safely experienced by the system may be obtained. This procedure is outlined in appendix A. As an example of this procedure, the time response to a triangular input was calculated by using Duhamel's integral with the response to a unit impulse obtained by the method of this paper for the multiple-mode system shown in figure 4. The comparison of the calculated time response with the exact time response for the arbitrary input shown in figure 7 serves to demonstrate the procedure and the accuracy. The time required to go through the operations with a desk-type computing machine is about 1 hour.

The evaluation of the Fourier transform of a time function is also useful for many purposes. One of the most important uses is the determination of the frequency response of input-output data without knowledge of the transfer function involved. The frequency response is simply the Fourier transform of the output divided by the Fourier transform of the input. The comparison shown in figure 8(c) of a frequency response obtained by using the method of this paper to evaluate the Fourier transform of the input and output time histories shown in figures 8(a) and 8(b) with the exact frequency response indicates that, for frequencies below the bottoming frequency of the input (about 12 radians per second), the results are very good. As would be expected the results above the bottoming frequency were poor since, at frequencies greater than the bottoming frequency of the input, the frequency content of the input is very small and in such cases the values obtained for a frequency response are unreliable.

The time required to calculate the frequency response shown in figure 8(c) with a desk-type computing machine is approximately 2.5 hours.

CONCLUDING REMARKS

A simple and rapid method is presented for the determination of the time response to a unit impulse from frequency-response data. Also included is a similar method of evaluating the Fourier transform of a function of time. The methods are applicable to linear functions for which Fourier transforms exist and give accurate results within the limitations of the values chosen for the variables.

Tables are presented which facilitate the necessary computations of the methods. These tables reduce the amount of labor and thus the time required for the necessary computations. Equations are presented which can be used to obtain greater accuracy or to extend the range of the variables. Suggested procedures for performing the necessary computations of the methods are illustrated and outlined step by step.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., September 22, 1955.

APPENDIX

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

Duhamel's integral may be stated as

$$x(t) = \delta(t)\ A(0) + \int_0^t h(t)\ \delta(t - \tau)\ d\tau \tag{A1}$$

where $A(0)$ is the indicial admittance at $t = 0$, $\delta(t)$ is the time history of the input, and $h(t)$ is the response of the system to a unit impulse. For the usual aircraft inputs, $A(0) = 0$ and

$$x(t) = \int_0^t h(t)\ \delta(t - \tau)\ d\tau \tag{A2}$$

This equation may be expressed numerically as

$$x\left(t_i\right) = \Delta t \sum_{j=0}^{i} C_{ij}\ h\left(t_j\right)\ \delta\left(t_{i-j}\right) \qquad j = 0, 1, 2, 3, \ldots i \tag{A3}$$

where

Δt incremental time interval

$\delta(t)$ time history of the arbitrary input

$h(t)$ time response to a unit impulse input

C_{ij} integrating matrix given in table V

Equation (A3) represents $i + 1$ equations. In order to illustrate its use, consider an arbitrary input $\delta(t)$ and a system with a known

response to a unit impulse $h(t)$. The response of this system to the arbitary input $\delta(t)$ is then computed for a time interval $\Delta t = 0.05$ second as follows:

$$x(0) = 0$$

$$x(0.05) = 0.05\left[0.5h(0)\ \delta(0.05) + 0.5h(0.05)\ \delta(0)\right] \tag{A4}$$

$$\begin{aligned} x(0.1) = 0.05\Big[&0.33333h(0)\ \delta(0.1) + \\ &1.333333h(0.05)\ \delta(0.05) + 0.33333h(0.1)\ \delta(0)\Big] \end{aligned} \tag{A5}$$

$$\begin{aligned} x(0.15) = 0.05\Big[&0.375h(0)\ \delta(0.15) + 1.125h(0.05)\ \delta(0.1) + \\ &1.125h(0.1)\ \delta(0.05) + 0.375h(0.15)\ \delta(0)\Big] \end{aligned} \tag{A6}$$

$$\begin{aligned} x(0.2) = 0.05\Big[&0.311h(0)\ \delta(0.2) + \\ &1.422h(0.05)\ \delta(0.15) + 0.533h(0.1)\ \delta(0.1) + \\ &1.422h(0.15)\ \delta(0.05) + 0.311h(0.2)\ \delta(0)\Big] \end{aligned} \tag{A7}$$

and so forth. The integrating matrix as presented herein has been found to be very accurate.

REFERENCES

1. Brown, Gordon S., and Campbell, Donald P.: Principles of Servomechanisms. John Wiley and Sons, Inc., 1948; Fourth printing corrected 1950, pp. 334-336.

2. Milne, William Edmund: Numerical Calculus. Princeton Univ. Press, 1949, pp. 122-124.

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$ USED TO CALCULATE THE TIME RESPONSE TO A UNIT IMPULSE FROM FREQUENCY-RESPONSE DATA AND TO CALCULATE THE REAL PART OF THE FOURIER TRANSFORM OF A TIME FUNCTION

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -														
	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	0.14
1	1.0000	0.9999	0.9997	0.9994	0.9989	0.9984	0.9976	0.9967	0.9957	0.9947	0.9933	0.9920	0.9904	0.9888	0.9869
2	1.0000	.9995	.9981	.9958	.9925	.9884	.9832	.9772	.9702	.9625	.9537	.9441	.9337	.9223	.9101
3	1.0000	.9988	.9949	.9887	.9798	.9687	.9547	.9386	.9201	.8992	.8761	.8508	.8233	.7939	.7623
4	1.0000	.9975	.9901	.9779	.9608	.9390	.9126	.8816	.8464	.8069	.7635	.7165	.6659	.6120	.5552
5	1.0000	.9959	.9837	.9637	.9356	.9001	.8572	.8074	.7510	.6886	.6206	.5476	.4702	.3891	.3048
6	1.0000	.9939	.9758	.9459	.9044	.8522	.7895	.7173	.6365	.5480	.4528	.3523	.2476	.1399	.0307
7	1.0000	.9916	.9663	.9248	.8675	.7958	.7105	.6132	.5056	.3897	.2670	.1400	.0108	-.1186	-.2458
8	1.0000	.9888	.9552	.9003	.8251	.7314	.6212	.4972	.3619	.2187	.0706	-.0789	-.2267	-.3691	-.5031
9	1.0000	.9856	.9427	.8726	.7774	.6597	.5231	.3713	.2090	.0407	-.1286	-.2942	-.4511	-.5949	-.7213
10	1.0000	.9820	.9286	.8418	.7246	.5814	.4173	.2383	.0507	-.1386	-.3227	-.4952	-.6496	-.7806	-.8836
11	1.0000	.9780	.9130	.8079	.6673	.4974	.3056	.1005	-.1089	-.3134	-.5040	-.6724	-.8109	-.9139	-.9765
12	1.0000	.9737	.8960	.7711	.6056	.4083	.1895	-.0391	-.2657	-.4781	-.6651	-.8171	-.9259	-.9857	-.9936
13	1.0000	.9689	.8775	.7316	.5401	.3152	.0707	-.1781	-.4156	-.6274	-.7998	-.9225	-.9876	-.9913	-.9334
14	1.0000	.9638	.8576	.6894	.4712	.2189	-.0492	-.3135	-.5551	-.7563	-.9025	-.9833	-.9928	-.9303	-.8004
15	1.0000	.9582	.8364	.6447	.3992	.1205	-.1683	-.4428	-.6803	-.8609	-.9693	-.9968	-.9410	-.8067	-.6051
16	1.0000	.9523	.8138	.5977	.3247	.0208	-.2850	-.5634	-.7881	-.9376	-.9974	-.9622	-.8353	-.6289	-.3626
17	1.0000	.9460	.7899	.5486	.2480	-.0791	-.3977	-.6732	-.8758	-.9840	-.9858	-.8813	-.6818	-.4088	-.0920
18	1.0000	.9394	.7647	.4976	.1699	-.1782	-.5045	-.7696	-.9412	-.9987	-.9349	-.7579	-.4891	-.1612	.1859
19	1.0000	.9323	.7384	.4447	.0907	-.2755	-.6041	-.8510	-.9825	-.9862	-.8467	-.5979	-.2684	.0971	.4492
20	1.0000	.9249	.7108	.3902	.0107	-.3700	-.6952	-.9158	-.9987	-.9317	-.7247	-.4091	-.0323	.3490	.6777
21	1.0000	.9171	.6821	.3342	-.0691	-.4609	-.7761	-.9625	-.9893	-.8523	-.5739	-.2006	.2056	.5775	.8533
22	1.0000	.9090	.6523	.2771	-.1486	-.5471	-.8459	-.9906	-.9547	-.7452	-.4001	.0176	.4318	.7670	.9624
23	1.0000	.9004	.6215	.2190	-.2271	-.6279	-.9036	-.9992	-.8958	-.6142	-.2104	.2349	.6332	.9051	.9966
24	1.0000	.8916	.5897	.1601	-.3042	-.7024	-.9481	-.9882	-.8139	-.4633	-.0124	.4409	.7983	.9822	.9531
25	1.0000	.8823	.5569	.1006	-.3794	-.7699	-.9791	-.9579	-.7112	-.2974	.1862	.6256	.9176	.9934	.8354
26	1.0000	.8727	.5233	.0408	-.4520	-.8292	-.9961	-.9089	-.5904	-.1219	.3776	.7802	.9843	.9379	.6527
27	1.0000	.8628	.4889	-.0192	-.5218	-.8817	-.9986	-.8421	-.4545	.0575	.5537	.8973	.9947	.8192	.4191
28	1.0000	.8525	.4536	-.0791	-.5883	-.9253	-.9869	-.7588	-.3070	.2351	.7081	.9710	.9479	.6455	.1529
29	1.0000	.8419	.4176	-.1388	-.6510	-.9563	-.9608	-.6607	-.1516	.4051	.8356	.9979	.8470	.4284	-.1253
30	1.0000	.8309	.3809	-.1979	-.7096	-.9825	-.9211	-.5496	.0076	.5620	.9220	.9767	.6973	.1825	-.3936
31	1.0000	.8196	.3436	-.2563	-.7636	-.9946	-.8681	-.4278	.1666	.7007	.9826	.9085	.5078	-.0762	-.6313
32	1.0000	.8080	.3058	-.3138	-.8127	-.9978	-.8026	-.2976	.3214	.8167	.9978	.7964	.2891	-.3293	-.8199
33	1.0000	.7961	.2675	-.3702	-.8566	-.9950	-.7255	-.1616	.4680	.9064	.9752	.6460	.0539	-.5601	-.9446
34	1.0000	.7838	.2288	-.4252	-.8950	-.9794	-.6380	-.0224	.6026	.9668	.9142	.4644	-.1845	-.7535	-.9957
35	1.0000	.7712	.1896	-.4787	-.9278	-.9502	-.5413	.1172	.7218	.9960	.8147	.2604	-.4122	-.8961	-.9693
36	1.0000	.7584	.1502	-.5304	-.9546	-.9174	-.4368	.2546	.8226	.9930	.6807	.0439	-.6163	-.9785	-.8673
37	1.0000	.7452	.1106	-.5803	-.9752	-.8740	-.3261	.3869	.9024	.9580	.5246	-.1747	-.7851	-.9951	-.6978
38	1.0000	.7317	.0707	-.6281	-.9897	-.8196	-.2107	.5117	.9591	.8918	.3494	-.3849	-.9089	-.9449	-.4740
39	1.0000	.7179	.0308	-.6736	-.9978	-.7616	-.0922	.6264	.9913	.7970	.1508	-.5766	-.9806	-.8313	-.2132
40	1.0000	.7038	-.0092	-.7167	-.9995	-.6884	.0276	.7289	.9982	.6763	-.0440	-.7405	-.9961	-.6617	.0642
41	1.0000	.6895	-.0492	-.7572	-.9929	-.6156	.1470	.8171	.9796	.5339	-.2453	-.8687	-.9544	-.4477	.3365
42	1.0000	.6749	-.0891	-.7950	-.9804	-.5314	.2642	.8894	.9360	.3741	-.4313	-.9550	-.8581	-.2036	.5827
43	1.0000	.6600	-.1288	-.8299	-.9665	-.4483	.3778	.9442	.8684	.2023	-.5992	-.9952	-.7126	.0542	.7835
44	1.0000	.6448	-.1684	-.8619	-.9430	-.3539	.4858	.9806	.7787	.0240	-.7462	-.9875	-.5262	.3083	.9232
45	1.0000	.6294	-.2077	-.8907	-.9134	-.2568	.5868	.9978	.6691	-.1552	-.8650	-.9322	-.3097	.5418	.9910
46	1.0000	.6137	-.2466	-.9164	-.8780	-.1616	.6795	.9954	.5424	-.3293	-.9461	-.8319	-.0753	.7388	.9817
47	1.0000	.5978	-.2852	-.9387	-.8370	-.0623	.7623	.9736	.4019	-.4928	-.9905	-.6916	.1632	.8861	.8959
48	1.0000	.5817	-.3233	-.9577	-.7908	.0376	.8342	.9328	.2510	-.6404	-.9955	-.5179	.3926	.9740	.7403
49	1.0000	.5653	-.3609	-.9732	-.7393	.1371	.8941	.8736	.0938	-.7672	-.9608	-.3193	.5993	.9962	.5271
50	1.0000	.5487	-.3979	-.9852	-.6832	.2353	.9411	.7974	-.0658	-.8693	-.8877	-.1052	.7716	.9516	.2728

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$.- Continued

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -														
	0.15	0.16	0.17	0.18	0.19	0.20	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29
1	0.9851	0.9830	0.9808	0.9785	0.9761	0.9736	0.9709	0.9680	0.9651	0.9620	0.9588	0.9557	0.9521	0.9486	0.9448
2	.8971	.8832	.8685	.8531	.8368	.8198	.8021	.7836	.7644	.7446	.7241	.7030	.6812	.6588	.6358
3	.7290	.6937	.6567	.6182	.5782	.5367	.4940	.4499	.4049	.3588	.3120	.2645	.2164	.1678	.1188
4	.4957	.4338	.3699	.3041	.2371	.1688	.0999	.0306	-.0389	-.1080	-.1764	-.2439	-.3100	-.3746	-.4370
5	.2182	.1298	.0406	-.0489	-.1380	-.2257	-.3115	-.3947	-.4745	-.5504	-.6216	-.6879	-.7481	-.8023	-.8499
6	-.0788	-.1873	-.2934	-.3958	-.4932	-.5846	-.6688	-.7446	-.8115	-.8684	-.9147	-.9499	-.9734	-.9851	-.9848
7	-.3688	-.4854	-.5937	-.6918	-.7781	-.8512	-.9098	-.9528	-.9798	-.9902	-.9838	-.9609	-.9216	-.8669	-.7977
8	-.6258	-.7342	-.8260	-.8992	-.9521	-.9834	-.9927	-.9795	-.9444	-.8882	-.8120	-.7178	-.6076	-.4839	-.3495
9	-.8270	-.9085	-.9639	-.9913	-.9901	-.9604	-.9030	-.8195	-.7127	-.5854	-.4415	-.2850	-.1206	.0470	.2129
10	-.9542	-.9905	-.9913	-.9563	-.8869	-.7857	-.6564	-.5035	-.3327	-.1503	.0372	.2231	.4007	.5635	.7057
11	-.9962	-.9720	-.9052	-.7987	-.6572	-.4870	-.2956	-.0916	.1163	.3187	.5068	.6724	.8079	.9078	.9677
12	-.9493	-.8548	-.7155	-.5387	-.3337	-.1114	.1164	.3378	.5412	.7157	.8522	.9437	.9852	.9749	.9131
13	-.8175	-.6508	-.4440	-.2097	.0374	.2818	.5084	.7030	.8536	.9510	.9891	.9658	.8822	.7441	.5599
14	-.6127	-.3808	-.1215	.1463	.4032	.6305	.8118	.9340	.9885	.9713	.8837	.7324	.5281	.2860	.0237
15	-.3532	-.0720	.2149	.4835	.7114	.8796	.9742	.9872	.9180	.7721	.5620	.3055	.0237	-.2595	-.5204
16	-.0621	.2440	.5266	.7587	.9182	.9899	.9673	.8523	.6566	.3984	.1027	-.2022	-.4874	-.7256	-.8943
17	.2345	.5353	.7780	.9366	.9939	.9439	.7922	.5552	.2586	-.0653	-.3817	-.6565	-.8599	-.9702	-.9756
18	.5102	.7723	.9405	.9944	.9279	.7489	.4794	.1522	-.1930	-.5143	-.7727	-.9372	-.9876	-.9183	-.7379
19	.7403	.9308	.9952	.9248	.7295	.4356	.0833	-.2798	-.6045	-.8470	-.9745	-.9701	-.8343	-.5859	-.2588
20	.9042	.9948	.9360	.7366	.4270	.0536	-.3273	-.6585	-.8907	-.9883	-.9377	-.7466	-.4436	-.0745	.3049
21	.9875	.9578	.7696	.4539	.0636	-.3369	-.6810	-.9118	-.9912	-.9062	-.6714	-.3256	.0734	.4596	.7689
22	.9824	.8235	.5150	.1131	-.3089	-.6742	-.9164	-.9913	-.8858	-.6194	-.2406	.1814	.5695	.8534	.9815
23	.8897	.6058	.2016	-.2422	-.6374	-.9051	-.9924	-.8820	-.5964	-.1925	.2491	.6404	.9035	.9864	.8729
24	.7175	.3264	-.1349	-.5665	-.8748	-.9931	-.8959	-.6047	-.1829	.2779	.6777	.9302	.9804	.8181	.4789
25	.4811	.0139	-.4560	-.8182	-.9874	-.9242	-.6438	-.2122	.2686	.6855	.9405	.9741	.7783	.3999	-.0717
26	.2005	-.2999	-.7249	-.9650	-.9592	-.7096	-.2796	.2207	.6642	.9381	.9723	.7604	.3547	-.1405	-.5989
27	-.0961	-.5834	-.9108	-.9880	-.7942	-.3820	.1330	.6116	.9218	.9787	.7667	.3457	-.1698	-.6380	-.9302
28	-.3843	-.8076	-.9924	-.8844	-.5158	.0044	.5226	.8860	.9877	.7982	.3759	-.1602	-.6461	-.9405	-.9573
29	-.6378	-.9498	-.9604	-.6674	-.1638	.3911	.8214	.9916	.8484	.4372	-.1119	-.6240	-.9384	-.9558	-.6713
30	-.8358	-.9956	-.8185	-.3648	.2115	.7151	.9773	.9083	.5326	-.0226	-.5685	-.9226	-.9637	-.6791	-.1656
31	-.9586	-.9402	-.5828	-.0155	.5566	.9256	.9634	.6520	.1061	-.4773	-.8897	-.9775	-.7147	-.1949	.3941
32	-.9978	-.7895	-.2803	.3358	.8224	.9946	.7820	.2715	-.3426	-.8241	-.9883	-.7739	-.2624	.3488	.8250
33	-.9422	-.5586	.0541	.6441	.9708	.8992	.4648	-.1607	-.7198	-.9847	-.8463	-.3657	.2647	.7859	.9860
34	-.8078	-.2709	.3825	.8698	.9807	.6706	.0666	-.5623	-.9475	-.9227	-.5021	.1391	.7163	.9831	.8245
35	-.6006	.0442	.6669	.9840	.8508	.3292	-.3430	-.8568	-.9782	-.6522	-.0251	.6073	.9642	.8798	.3934
36	-.3358	.3549	.8751	.9720	.5993	-.0624	-.6930	-.9880	-.8055	-.2342	.4483	.9148	.9376	.5078	-.1663
37	-.0464	.6295	.9831	.8355	.2624	-.4435	-.9226	-.9311	-.4655	.2366	.8167	.9805	.6442	-.0193	-.6718
38	.2524	.8398	.9784	.5917	-.1119	-.7561	-.9918	-.6967	-.0285	.6540	.9847	.7870	.1675	-.5406	-.9574
39	.5239	.9656	.8618	.2722	-.4703	-.9469	-.8887	-.3297	.4143	.9235	.9127	.3855	-.3568	-.8967	-.9300
40	.7509	.9930	.6466	-.0823	-.7615	-.9891	-.6310	.1002	.7710	.9844	.6111	-.1180	-.7797	-.9788	-.5983
41	.9106	.9195	.3573	-.4262	-.9442	-.8749	-.2637	.5110	.9674	.8227	.1686	-.5903	-.9807	-.7620	-.0709
42	.9908	.7527	.0271	-.7154	-.9921	-.6227	.1495	.8245	.9627	.4838	-.3210	-.9064	-.9025	-.3124	.4796
43	.9788	.5095	-.3062	-.9129	-.8985	-.2764	.5367	.9808	.7579	.0202	-.7299	-.9831	-.5675	.2327	.8733
44	.8790	.2146	-.6045	-.9934	-.6767	.1224	.8306	.9503	.3955	-.4393	-.9593	-.7997	-.0710	.7067	.9814
45	.7063	-.1022	-.8336	-.9466	-.3583	.4951	.9801	.7389	-.0492	-.7995	-.9589	-.4051	.4456	.9647	.7684
46	.4665	-.4085	-.9671	-.7784	.0111	.7910	.9592	.3866	-.4835	-.9791	-.7181	.0967	.8356	.9281	.3042
47	.1854	-.6735	-.9900	-.5103	.3790	.9625	.7716	-.0393	-.8174	-.9373	-.3038	.5730	.9876	.6079	-.2596
48	-.1122	-.8699	-.8996	-.1768	.6928	.9821	.4499	-.4577	-.9813	-.6838	.1850	.8977	.8587	.1021	-.7384
49	-.3997	-.9782	-.7061	.1792	.9078	.8466	.0500	-.7890	-.9412	-.2756	.6284	.9851	.4854	-.4350	-.9757
50	-.6516	-.9870	-.4318	.5124	.9932	.5775	-.3586	-.9699	-.7054	.1948	.9180	.8121	-.0261	-.8391	-.8939

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$.- Continued

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -														
	0.30	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40	0.41	0.42	0.43	0.44
1	0.9411	0.9371	0.9331	0.9273	0.9247	0.9203	0.9158	0.9112	0.9065	0.9016	0.8967	0.8916	0.8865	0.8813	0.8758
2	.6123	.5882	.5638	.5378	.5134	.4875	.4612	.4346	.4076	.3804	.3528	.3249	.2969	.2686	.2403
3	.0697	.0205	-.0286	-.0775	-.1263	-.1746	-.2223	-.2694	-.3156	-.3608	-.4051	-.4483	-.4901	-.5307	-.5697
4	-.4973	-.5549	-.6099	-.6604	-.7098	-.7546	-.7954	-.8325	-.8651	-.8934	-.9173	-.9365	-.9512	-.9612	-.9662
5	-.8906	-.9238	-.9496	-.9658	-.9776	-.9797	-.9738	-.9601	-.9385	-.9094	-.8730	-.8297	-.7797	-.7234	-.6615
6	-.9727	-.9487	-.9135	-.8656	-.8103	-.7440	-.6687	-.5856	-.4955	-.3996	-.2992	-.1954	-.0896	.0171	.1232
7	-.7151	-.6205	-.5158	-.4018	-.2827	-.1584	-.0317	.0952	.2202	.3412	.4561	.5629	.6601	.7457	.8185
8	-.2077	-.0613	.0860	.2307	.3707	.5017	.6211	.7262	.8148	.8847	.9348	.9635	.9708	.9560	.9198
9	.3723	.5206	.6538	.7663	.8592	.9258	.9655	.9774	.9609	.9168	.8464	.7518	.6358	.5018	.3536
10	.8222	.9089	.9628	.9801	.9655	.9145	.8307	.7173	.5782	.4188	.2446	.0622	-.1220	-.3013	-.4692
11	.9849	.9588	.8907	.7822	.6422	.4731	.2836	.0820	-.1232	-.3219	-.5056	-.6673	-.7987	-.8949	-.9515
12	.8036	.6518	.4660	.2557	.0333	-.1908	-.4043	-.5961	-.7564	-.8760	-.9491	-.9722	-.9442	-.8664	-.7433
13	.3415	.1021	-.1430	-.3781	-.5904	-.7650	-.8915	-.9625	-.9733	-.9237	-.8169	-.6592	-.4618	-.2357	.0043
14	-.2399	-.4855	-.6955	-.8532	-.9516	-.9794	-.9361	-.8254	-.6547	-.4377	-.1892	.0722	.3278	.5589	.7487
15	-.7375	-.8924	-.9727	-.9699	-.8894	-.7332	-.5161	-.2566	.0238	.3014	.5532	.7581	.8993	.9650	.9499
16	-.9774	-.9672	-.8649	-.6793	-.4316	-.1421	.1602	.4465	.6895	.8662	.9601	.9622	.8727	.7004	.4617
17	-.8759	-.6819	-.4147	-.1033	.2182	.5157	.7569	.9160	.9758	.9303	.7846	.5547	.2657	-.0512	-.3615
18	-.4685	-.1428	.1995	.5160	.7710	.9311	.9779	.9063	.7250	.4564	.1332	-.2053	-.5239	-.7671	-.9224
19	.1027	.4495	.7349	.9186	.9807	.9085	.7135	.4226	.0753	-.2813	-.5991	-.8348	-.9572	-.9498	-.8139
20	.6379	.8744	.9794	.9354	.7541	.4587	.0949	-.2821	-.6159	-.8564	-.9679	-.9338	-.7598	-.4722	-.1147
21	.9503	.9739	.8361	.5593	.1896	-.2069	-.5708	-.8393	-.9681	-.9363	-.7496	-.4392	-.0571	.3335	.6677
22	.9308	.7108	.3620	-.0518	-.4554	-.7752	-.9532	-.9576	-.7875	-.4748	-.0766	.3344	.6836	.9075	.9656
23	.5861	.1831	-.2554	-.6411	-.9003	-.9788	-.8624	-.5749	-.1736	.2611	.6428	.8956	.9696	.8506	.5628
24	.0366	-.4127	-.7718	-.9611	-.9447	-.7222	-.3436	.1085	.5359	.8461	.9724	.8875	.6108	.2024	-.2485
25	-.5256	-.8549	-.9826	-.8775	-.5690	-.1258	.3459	.7351	.9505	.9419	.7121	.3154	-.1543	-.5865	-.8794
26	-.9056	-.9789	-.8046	-.4253	.0600	.5293	.8634	.9772	.8420	.4932	.0222	-.4572	-.8168	-.9677	-.8721
27	-.9678	-.7385	-.3081	.2054	.6622	.9361	.9527	.7082	.2701	-.2408	-.6870	-.9392	-.9360	-.6763	-.2319
28	-.6900	-.2232	.3103	.7500	.9698	.9015	.5689	.0687	-.4504	-.8355	-.9736	-.8243	-.4327	.0852	.5765
29	-.1746	.3752	.8060	.9794	.8460	.4437	-.0973	-.6067	-.9230	-.9471	-.6703	-.1855	.3584	.7874	.9666
30	.4020	.8339	.9825	.7975	.3458	-.2241	-.7152	-.9647	-.8878	-.5112	.0377	.5712	.9111	.9424	.6552
31	.8387	.9822	.7703	.2806	-.3082	-.7848	-.9780	-.8181	-.3639	.2203	.7218	.9649	.8579	.4422	-.1316
32	.9851	.7649	.2531	-.3540	-.8250	-.9788	-.7554	-.2437	.3603	.8245	.9724	.7453	.2342	-.3654	-.8230
33	.7835	.2628	-.3642	-.8401	-.9750	-.7108	-.1578	.4583	.8861	.9519	.6281	.0521	-.5454	-.9189	-.9172
34	.3097	-.3370	-.8374	-.9733	-.6959	-.1068	.5181	.9205	.9243	.5290	-.0931	-.6742	-.9622	-.8338	-.3457
35	-.2730	-.8115	-.9792	-.6977	-.0998	.5437	.9368	.9013	.4539	-.1997	-.7587	-.9720	-.7390	-.1690	.4766
36	-.7594	-.9838	-.7333	-.1290	.5358	.9407	.8905	.4106	-.2664	-.8130	-.9659	-.6521	-.0245	.6132	.9531
37	-.9777	-.7900	-.1972	.4947	.9332	.8944	.4022	-.2948	-.8400	-.9562	-.5872	.0823	.7064	.9692	.7379
38	-.8605	-.3020	.4169	.9109	.9154	.4296	-.2858	-.8460	-.9514	-.5466	.1496	.7644	.9675	.6514	-.0137
39	-.4412	.2983	.8660	.9445	.4904	-.2395	-.8319	-.9547	-.5392	.1791	.7949	.9607	.5851	-.1201	-.7547
40	.1378	.7876	.9724	.5813	-.1528	-.7937	-.9651	-.5641	.1697	.8012	.9571	.5464	-.1874	-.8074	-.9481
41	.6634	.9837	.6938	-.0259	-.7279	-.9784	-.6192	.1217	.7853	.9601	.5420	-.2161	-.8344	-.9334	-.4533
42	.9580	.8137	.1407	-.6224	-.9793	-.6990	.0341	.7438	.9686	.5638	-.2094	-.8405	-.9265	-.4107	.3695
43	.9186	.3408	-.4682	-.9574	-.7950	-.0909	.6704	.9768	.6189	-.1594	-.8228	-.9307	-.4024	.3967	.9250
44	.5564	-.2591	-.8917	-.8903	-.2571	.5531	.9740	.6989	-.0723	-.7896	-.9458	-.4293	.3885	.9293	.8092
45	.0046	-.7624	-.9623	-.4493	.3953	.9487	.7941	.0555	-.7231	-.9634	-.4895	.3440	.9219	.8158	.1063
46	-.5533	-.9819	-.6520	.1804	.8717	.8889	.2201	-.6178	-.9758	-.5801	.2612	.8995	.8422	.1352	-.6738
47	-.9168	-.8360	-.0837	.7343	.9605	.4144	-.4641	-.9669	-.6916	.1375	.8547	.8833	.2024	-.6393	-.9650
48	-.9602	-.3788	.5178	.9798	.6218	-.2558	-.9170	-.8103	-.0267	.7707	.9298	.3059	-.5720	-.9695	-.5558
49	-.6680	.2193	.9143	.8138	.0067	-.8049	-.9146	-.2299	.6521	.9664	.4409	-.4661	-.9659	-.6257	.2567
50	-.1425	.7358	.9490	.3059	-.6123	-.9754	-.4582	.4699	.9731	.5976	-.3154	-.9418	-.7175	.1529	.8829

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$.- Continued

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -														
	0.45	0.46	0.47	0.48	0.49	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
1	0.8704	0.8648	0.8591	0.8533	0.8474	0.8415	0.8102	0.7767	0.7412	0.7039	0.6650	0.6247	0.5834	0.5410	0.4980
2	.2117	.1830	.1543	.1254	.0966	.0678	-.0752	-.2138	-.3447	-.4646	-.5709	-.6612	-.7337	-.7869	-.8201
3	-.6072	-.6430	-.6771	-.7094	-.7398	-.7682	-.8784	-.9317	-.9256	-.8619	-.7458	-.5861	-.3943	-.1835	.0322
4	-.9666	-.9620	-.9530	-.9391	-.9208	-.8979	-.7117	-.4614	-.1505	.1717	.4654	.6955	.8353	.8702	.7993
5	-.5945	-.5227	-.4470	-.3679	-.2860	-.2021	.2137	.5973	.8451	.9202	.8116	.5455	.1791	-.2120	-.5490
6	.2275	.3287	.4257	.5172	.6021	.6795	.9246	.8943	.6027	.1412	-.3506	-.7273	-.8814	-.7739	-.4443
7	.8773	.9211	.9491	.9611	.9569	.9364	.6151	.0508	-.5227	-.8722	-.8612	-.5031	.0481	.5636	.8362
8	.8632	.7872	.6939	.5853	.4638	.3324	-.3666	-.8575	-.8823	-.4377	.2287	.7567	.8690	.5178	-.0964
9	.1958	.0328	-.1307	-.2898	-.4401	-.5773	-.9477	-.6722	.0506	.7234	.8936	.4589	-.2720	-.7989	-.7739
10	-.6197	-.7475	-.8481	-.9177	-.9542	-.9562	-.4932	.3703	.9094	.6836	-.1023	-.7835	-.7989	-.1548	.5968
11	-.9663	-.9386	-.8696	-.7627	-.6229	-.4560	.5003	.9406	.4359	-.4910	-.9080	-.4131	.4779	.8693	.3881
12	-.5816	-.3896	-.1778	.0427	.2603	.4634	.9470	.3114	-.6762	-.8505	-.0262	.8076	.6758	-.2402	-.8477
13	.2433	.4664	.6599	.8117	.9129	.9568	.3589	-.7149	-.7976	.2019	.9043	.3660	-.6520	-.7601	.1600
14	.8840	.9547	.9563	.8884	.7567	.5705	-.6215	-.8295	.2494	.9192	.1541	-.8290	-.5078	.5857	.7442
15	.8558	.6904	.4680	.2073	-.0698	-.3403	-.9227	.1138	.9311	.1105	-.8825	-.3176	.7829	.4940	-.6412
16	.1799	-.1182	-.4041	-.6507	-.8345	-.9382	-.2155	.9119	.2487	-.8816	-.2790	.8475	.3061	-.8101	-.3296
17	-.6321	-.8337	-.9448	-.9536	-.8598	-.6735	.7271	.5471	-.7980	-.4102	.8431	.2681	-.8617	-.1259	.8544
18	-.9657	-.8918	-.7103	-.4432	-.1233	.2104	.8752	-.5154	-.6750	.7421	.3983	-.8632	-.0840	.8672	-.2236
19	-.5685	-.2470	.1070	.4452	.7224	.9007	.0669	-.9208	.4365	.6625	-.7867	-.2177	.8833	-.2690	-.7098
20	.2589	.5927	.8364	.9539	.9281	.7631	-.8145	-.1518	.9092	-.5169	-.5096	.8759	-.1445	-.7450	.6820
21	.8904	.9650	.8797	.6490	.3116	-.0763	-.8068	.8106	.0499	-.8382	.7152	.1657	-.8461	.6069	.2693
22	.8481	.5766	.2012	-.2095	-.5810	-.8455	.0835	.7393	-.8825	.2329	.6100	-.8857	.3617	.4697	-.8562
23	.1639	-.2664	-.6423	-.8894	-.9589	-.8374	.8816	-.2749	-.5220	.9170	-.6289	-.1139	.7532	-.8203	.2842
24	-.6443	-.8994	-.9589	-.8105	-.4873	-.0594	.7162	-.9385	.6039	.0789	-.6990	.8923	-.5558	-.0969	.6724
25	-.9649	-.8234	-.4888	-.0404	.4151	.7732	-.2318	-.4053	.8444	-.8902	.5292	.0627	-.6100	.8644	-.7190
26	-.5549	-.0982	.3824	.7642	.9507	.8943	-.9262	.6432	-.1522	-.3840	.7747	-.8940	.7124	-.2957	-.2079
27	.2764	.7043	.9398	.9170	.6431	.1940	-.6079	.8733	-.9247	.7619	-.4181	-.0111	.4261	-.7303	.8540
28	.8964	.9517	.7262	.2876	-.2343	-.6842	.3727	-.0135	-.3451	.6383	-.8345	.8965	-.8223	.6291	-.3444
29	.8379	.4488	-.0832	-.5871	-.9041	-.9357	.9498	-.8805	.7420	-.5401	.3001	-.0417	-.2155	.4435	-.6312
30	.1485	-.4079	-.8243	-.9610	-.7729	-.3229	.4856	-.6249	.7410	-.8251	.8773	-.8947	.8773	-.8281	.7500
31	-.6559	-.9430	-.8892	-.5152	.0430	.5840	-.5060	.4251	-.3442	.2603	-.1764	.0956	-.0117	-.0692	.1471
32	-.9629	-.7346	-.2245	.3700	.8208	.9534	-.9470	.9375	-.9344	.9153	-.9026	.8868	-.8741	.8614	-.8455
33	-.5421	.0529	.6243	.9397	.8715	.4495	-.3503	.2511	-.1422	.0493	.0500	-.1460	.2387	-.3248	.4011
34	.2897	.7987	.9610	.7078	.1500	-.4695	.6257	-.7547	.8464	-.8973	.9074	-.8800	.8117	-.7126	.5861
35	.9037	.9149	.5093	-.1287	-.7050	-.9574	.9196	-.7970	.5999	-.3561	.0796	.1981	-.4456	.6487	-.7828
36	.8309	.3098	-.3612	-.8548	-.9345	-.5630	.2094	.1762	-.5217	.7797	-.8958	.8691	-.6988	.4275	-.0796
37	.1330	-.5403	-.9344	-.8518	-.3360	.3462	-.7346	.9235	-.8812	.6169	-.2064	-.2476	.6275	-.8476	.8340
38	-.6706	-.9636	-.7411	-.1223	.5593	.9402	-.8706	.4955	.0512	-.5659	.8660	-.8546	.5369	-.0392	-.4580
39	-.9609	-.6272	.0594	.7109	.9600	.6669	-.0586	-.5670	.9089	-.8112	.3313	.2895	-.7648	.8562	-.5374
40	-.5267	.2027	.8116	.9386	.5101	-.2193	.8194	-.9052	.4363	.2910	-.8241	.8451	-.3408	-.3514	.8030
41	.3046	.8735	.8981	.3658	-.3915	-.8998	.7999	-.0898	-.6780	.9091	-.4438	-.3464	.8519	-.6935	.0197
42	.9088	.8556	.2477	-.5191	-.9465	-.7598	-.0893	.8432	-.7960	.0173	.7612	-.8176	.1213	.6627	-.8155
43	.8240	.1633	-.6059	-.9612	-.6628	.0857	-.8857	.6950	.2496	-.9027	.5531	.3945	-.8838	.3939	.5084
44	.1142	-.6578	-.9624	-.5835	.2080	.8475	-.7121	-.3332	.9308	-.3249	-.6841	.7921	.1086	-.8419	.4842
45	-.6781	-.9604	-.5294	.2920	.8947	.8329	.2412	-.9387	.2491	.7947	-.6484	-.4409	.8565	-.0122	-.8213
46	-.9608	-.5057	.3380	.9183	.7886	.0510	.9283	-.3474	-.7984	.5949	.5894	-.7677	-.3285	.8497	.0479
47	-.5147	.3475	.9280	.7614	-.0161	-.7781	.6023	.6894	-.6751	-.5909	.7318	.4857	-.7714	-.3767	.7934
48	.3209	.9269	.7567	-.0450	-.8065	-.8919	-.3819	.8469	.4372	-.7957	-.4859	.7392	.5273	-.6786	-.5610
49	.9137	.7754	-.0355	-.8130	-.8824	-.1856	-.9488	-.0756	.9090	.3204	-.8006	-.5289	.6355	.6850	-.4308
50	.8150	.0127	-.7985	-.8875	-.1764	.6914	-.4788	-.9017	.0492	.9047	.3726	-.7084	-.6910	.3673	.8395

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$.- Continued

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -															
	1.0	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75
1	0.4547	0.4111	0.3675	0.3242	0.2814	0.2394	0.1982	0.1582	0.1197	0.0825	0.0470	0.0134	-0.0182	-0.0478	-0.0751	-0.1002
2	-.8331	-.8261	-.8001	-.7562	-.6964	-.6230	-.5380	-.4445	-.3451	-.2427	-.1402	-.0402	.0547	.1422	.2205	.2880
3	.2387	.4231	.5742	.6834	.7456	.7589	.7239	.6455	.5307	.3888	.2305	.0670	-.0908	-.2330	-.3511	-.4390
4	.6344	.3987	.1244	-.1546	-.4034	-.5929	-.7024	-.7226	-.6549	-.5124	-.3163	-.0936	.1268	.3181	.4585	.5344
5	-.7667	-.8258	-.7205	-.4773	-.1507	.1911	.4800	.6611	.7035	.6061	.3956	.1200	-.1624	-.3951	-.5354	-.5618
6	.0037	.4352	.7233	.7909	.6261	.2869	-.1201	-.4727	-.6707	-.6647	-.4671	-.1463	.1972	.4623	.5768	.5178
7	.7636	.3863	-.1309	-.5763	-.7722	-.6503	-.2741	.1962	.5605	.6846	.5292	.1723	-.2315	-.5179	-.5799	-.4081
8	-.6392	-.8249	-.5691	-.0229	.5128	.7547	.5898	.1226	-.3855	-.6648	-.5807	-.1980	.2649	.5605	.5444	.2464
9	-.2316	.4467	.8011	.6064	.0152	-.5601	-.7368	-.4152	.1660	.6064	.6206	.2234	-.2975	-.5890	-.4728	-.0535
10	.8320	.3737	-.3739	-.7856	-.5354	.1423	.6728	.6283	.0728	-.5127	-.6482	-.2483	.3291	.6029	.3698	-.1468
11	-.4609	-.8248	-.3612	.4406	.7741	.3324	-.4163	-.7207	-.3031	.3893	.6627	.2728	-.3600	-.6015	-.2418	.3274
12	-.4484	.4592	.7984	.1986	-.6059	-.6742	.0406	.6750	.4984	-.2439	-.6639	-.2964	.3887	.5853	.0987	-.4670
13	.8341	.3615	-.5788	-.7054	.1191	.7484	.3467	-.5002	-.6358	.0830	.6518	.3205	-.4166	-.5543	.0515	.5472
14	-.2458	-.8237	-.1176	.7408	.4299	-.5244	-.6345	.2285	.7003	.0819	-.6258	-.3435	.4430	.5095	-.1983	-.5579
15	-.6295	.4703	.7177	-.2811	-.7527	.0913	.7412	.0865	-.6836	-.2422	.5891	.3659	-.4680	-.4518	.3318	.4976
16	.7687	.3483	-.7265	-.3663	.6813	.3771	-.6354	-.3849	.5878	.3884	-.5396	-.3876	.4913	.3829	-.4434	-.3742
17	-.0100	-.8223	.1377	.7692	-.2518	-.6954	.3478	.6095	-.4242	-.5119	.4795	.4088	-.5130	-.3044	.5256	.2031
18	-.7601	.4821	.5645	-.6588	-.3104	.7380	.0394	-.7171	.2115	.6059	-.4096	-.4292	.5330	.2182	-.5727	-.0063
19	.6440	.3371	-.8026	.1088	.7095	-.1939	-.4153	.6871	.0256	-.6645	.3316	.4488	-.5510	-.1266	.5820	-.1914
20	.2245	-.8217	.3804	.5136	-.7361	-.2503	.6723	-.5253	-.2598	.6846	-.2469	-.4677	.5673	.0319	-.5525	.3647
21	-.8308	.4927	.3553	-.7931	.3767	.4190	-.7370	.2627	.4639	-.6649	.1573	.4857	-.5816	.0637	.4864	-.4916
22	.4668	.3239	-.7989	.5441	.1809	-.7154	.5906	.0502	-.6144	.6066	-.0645	-.5030	.5939	-.1579	-.3880	.5561
23	.4421	-.8199	.5851	.0686	-.6447	.7270	-.2752	-.3536	.6940	-.5130	-.0295	.5194	-.6042	.2477	.2638	-.5499
24	-.8350	.5041	.1097	-.6364	.7695	-.4490	-.1190	.5891	-.6933	.3897	.1230	-.5348	.6125	-.3315	-.1220	.4739
25	.2528	.3102	-.7142	.7798	-.4899	-.0076	.4791	-.7116	.6125	-.2437	-.2140	.5493	-.6186	.4070	-.0277	-.3376
26	.6240	-.8170	.7306	-.4018	-.0519	.5602	-.7021	.6975	-.4610	.0836	.3007	-.5629	.6226	-.4724	.1757	.1584
27	-.7721	.5149	-.1460	-.2448	.5709	-.7298	.7241	-.5496	.2561	.0814	-.3814	.5755	-.6245	.5259	-.3120	.0409
28	.0177	.2967	-.5580	.7256	-.7835	.7104	-.5389	.2963	-.0217	-.2417	.4545	-.5871	.6243	-.5662	.4276	-.2350
29	.7563	-.8162	.8025	-.7240	.5863	-.4065	.1994	.0139	-.2179	.3879	-.5184	.5977	-.6220	.5924	-.5148	.3993
30	-.6489	.5276	-.3861	.2390	-.0893	-.0603	.1972	-.3214	.4273	-.5116	.5720	-.6073	.6175	-.6037	.5678	-.5128
31	-.2159	.2848	-.3476	.4075	-.4553	.5001	-.5372	.5673	-.5900	.6056	-.6141	.6158	-.6109	.6000	-.5831	.5611
32	.8297	-.8138	.7948	-.7789	.7599	-.7409	.7236	-.7043	.6845	-.6644	.6440	-.6233	.6023	-.5811	.5597	-.5382
33	-.4740	.5371	-.5870	.6303	-.6637	.6872	-.7028	.7062	-.6999	.6846	-.6609	.6296	-.5915	.5478	-.4991	.4468
34	-.4356	.2713	-.1032	-.0774	.2201	-.3618	.4809	-.5726	.6344	-.6650	.6647	-.6348	.5788	-.5007	.4054	-.2987
35	.8373	-.8119	.7100	-.5487	.3385	-.1071	-.1213	.3292	-.4956	.6068	-.6551	.6391	-.5641	.4411	-.2848	.1126
36	-.2625	.5451	-.7322	.7949	-.7221	.5349	-.2730	-.0226	.2996	-.5135	.6324	-.6422	.5475	-.3704	.1452	.0878
37	-.6181	.2616	.1514	-.5108	.7247	-.7486	.5892	-.2884	-.0688	.3902	-.5970	.6441	-.5291	.2905	.0040	-.2770
38	.7733	-.8074	.5514	-.1134	-.3451	.6658	-.7367	.5440	-.1698	-.2443	.5498	-.6449	.5087	-.2033	-.1529	.4311
39	-.0238	.5536	-.8039	.6608	-.2116	-.3165	.6734	-.6952	.3888	.0841	-.4914	.6447	-.4867	.1110	.2917	-.5304
40	-.7534	.2466	.3943	-.7690	.6576	-.1611	-.4173	.7130	-.5629	.0809	.4233	-.6433	.4630	-.0160	-.4111	.5622
41	.6518	-.8031	.3407	.3646	-.7606	.5730	.0418	-.5941	.6719	-.2412	-.3467	.6408	-.4377	-.0795	.5032	-.5226
42	.2096	.5643	-.7935	.2862	.4618	-.7554	.3457	.3612	-.7033	.3875	.2631	-.6372	.4110	.1730	-.5618	.4166
43	-.8255	.2348	.5916	-.7451	.0800	.6401	-.6342	-.0589	.6535	-.5113	-.1743	.6325	-.3828	-.2622	.5832	-.2576
44	.4777	-.8016	.0960	.7027	-.5808	-.2713	.7410	-.2546	-.5281	.6053	.0820	-.6267	.3533	.3448	-.5659	.0660
45	.4279	.5746	-.7053	-.1762	.7786	-.2074	-.6360	.5193	.3417	-.6643	.0120	.6198	-.3227	-.4187	.5110	.1341
46	-.8367	.2194	.7361	-.4439	-.5651	.6033	.3488	-.6843	-.1157	.6846	-.1057	-.6118	.2909	.4822	-.4221	-.3171
47	.2669	-.7983	-.1594	.7864	.0568	-.7591	.0382	.7181	-.1235	-.6652	.1973	.6028	-.2581	-.5336	.3053	.4598
48	.6145	.5866	-.5484	-.6040	.4813	.6131	-.4143	-.6141	.3485	.6071	-.2850	-.5928	.2245	.5716	-.1681	-.5441
49	-.7785	.2059	.8049	.0184	-.7667	-.2232	.6718	.3923	-.5333	-.5138	.3669	.5817	-.1901	-.5953	.0198	.5593
50	.0333	-.7945	-.3989	.5795	.6493	-.2555	-.7370	-.0951	.6564	.3906	-.4415	-.5696	.1551	.6041	.1298	-.5034

TABLE I.- TABULATION OF THE FUNCTION $\frac{\sin z \cos(2n - 1)z}{z}$.- Concluded

n	Value of function $\frac{\sin z \cos(2n - 1)z}{z}$ at values of z of -														
	1.80	1.85	1.90	1.95	2.0	2.05	2.10	2.15	2.20	2.25	2.30	2.35	2.40	2.45	2.50
1	-0.1229	-0.1432	-0.1610	-0.1764	-0.1892	-0.1996	-0.2075	-0.2131	-0.2163	-0.2172	-0.2160	-0.2128	-0.2075	-0.2005	-0.1918
2	.3434	.3861	.4158	.4324	.4365	.4290	.4111	.3839	.3492	.3088	.2644	.2180	.1712	.1257	.0830
3	-.4929	-.5117	-.4967	-.4514	-.3815	-.2937	-.1955	-.0946	.0016	.0870	.1567	.2074	.2389	.2474	.2389
4	.5407	.4818	.3700	.2231	.0622	-.0914	-.2193	-.3080	-.3502	-.3455	-.2996	-.2232	-.1296	-.0334	.0525
5	-.4768	-.3056	-.0886	.1276	.3002	.3988	.4106	.3416	.2137	.0586	-.0895	-.2018	-.2602	-.2599	-.2091
6	.3145	.0365	-.2298	-.4083	-.4546	-.3671	-.1832	.0342	.2189	.3208	.3197	.2282	.0841	-.0635	-.1711
7	-.0873	.2436	.4522	.4652	.2906	.0232	-.2309	-.3690	-.3482	-.1939	.0178	.1962	.2749	.2362	.1120
8	-.1580	-.4498	-.4855	-.2671	.0701	.3404	.4097	.2616	-.0052	-.2393	-.3237	-.2329	-.0358	.1516	.2347
9	.3706	.5193	.3158	-.0779	-.3860	-.4144	-.1704	.1593	.3512	.2947	.0549	-.1904	-.2812	-.1796	.0211
10	-.5070	-.4307	-.0141	.3795	.4342	.1362	-.2422	-.3893	-.2110	.1148	.3114	.2378	-.0132	-.2186	-.2227
11	.5382	.2119	-.2934	-.4735	-.1819	.2580	.4083	.1528	-.2215	-.3431	-.1247	.1845	.2789	.0981	-.1474
12	-.4586	.0717	.4784	.3081	-.1964	-.4327	-.1581	.2668	.3471	.0298	-.2834	-.2424	.0620	.2552	.1390
13	.2842	-.3335	-.4633	.0263	.4382	.2396	-.2532	-.3666	.0081	.3305	.1883	-.1785	-.2680	-.0029	.2263
14	-.0512	.4939	.2545	-.3462	-.3770	.1574	.4064	.0271	-.3521	-.1692	.2411	.2468	-.1088	-.2563	-.0106
15	-.1924	-.5044	.0606	.4764	.0542	-.4204	-.1453	.3450	.2083	-.2591	-.2424	.1724	.2490	-.0927	-.2324
16	.3963	.3615	-.3505	-.3454	.3061	.3260	-.2640	-.3036	.2240	.2784	-.1868	-.2511	.1524	.2217	-.1212
17	-.5183	-.1089	.4938	.0252	-.4545	.0456	.4041	-.1016	-.3461	.1417	.2843	-.1661	-.2223	.1754	.1636
18	.5334	-.1768	-.4307	.3089	.2879	-.3785	-.1323	.3850	-.0113	-.3382	.1230	.2552	-.1913	-.1563	.2140
19	-.4383	.4088	.1875	-.4737	.0780	.3895	-.2744	-.2071	.3530	.0008	-.3118	.1598	.1888	-.2336	-.0422
20	.2527	-.5166	.1340	.3788	-.3899	-.0693	.4013	-.2191	-.2057	.3378	-.0531	-.2592	.2244	.0691	-.2379
21	-.0149	.4675	-.3995	-.0763	.4317	-.3098	-.1191	.3826	-.2266	-.1433	.3237	-.1534	-.1496	.2594	-.0928
22	-.2259	-.2764	.4980	-.2681	-.1745	.4255	-.2845	-.0877	.3450	-.2774	-.0196	.2630	-.2505	.0276	.1853
23	.4201	.0012	-.3883	.4654	-.2036	-.1794	.3981	-.3123	.0146	.2602	-.3194	.1469	.1057	-.2491	.1979
24	-.5276	.2743	.1163	-.4078	.4407	-.2193	-.1059	.3381	-.3539	.1677	.0912	-.2666	.2690	-.1206	-.0730
25	.5261	-.4664	.2044	.1265	-.3725	.4315	-.2943	.0413	.2030	-.3310	.2989	-.1403	-.0587	.2042	-.2393
26	-.4160	.5169	-.4396	.2241	.0463	-.2768	.3945	-.3712	.2291	-.0282	-.1583	.2701	-.2793	.1967	-.0627
27	.2200	-.4103	.4911	-.4518	.3120	-.1133	-.0925	.2562	-.3438	.3429	-.2634	.1336	.0098	-.1308	.2038
28	.0214	.1792	-.3372	.4319	-.4541	.4070	-.3038	.1658	-.0178	-.1164	.2173	-.2734	.2810	-.2455	.1783
29	-.2584	.1065	.0424	-.1753	.2817	-.3546	.3903	-.3891	.3548	-.2938	.2147	-.1268	.0394	.0392	-.1026
30	.4421	-.3598	.2701	-.1774	.0859	.0007	-.0789	.1461	-.2003	.2402	-.2655	.2765	-.2741	.2601	-.2365
31	-.5345	.5038	-.4697	.4329	-.3940	.3538	-.3129	.2720	-.2316	.1925	-.1551	.1199	-.0873	.0578	-.0316
32	.5164	-.4947	.4729	-.4511	.4292	-.4075	.3858	-.3642	.3427	-.3214	.3003	-.2795	.2588	-.2386	.2186
33	-.3918	.3353	-.2784	.2220	-.1671	.1146	-.0653	.0199	.0210	-.0570	.0877	-.1130	.1326	-.1468	.1556
34	.1863	-.0741	-.0325	.1287	-.2107	.2757	-.3217	.3482	-.3556	.3454	-.3200	.2823	-.2356	.1838	-.1303
35	.0577	-.2096	.3298	-.4089	.4426	-.4316	.3808	-.2992	.1976	-.0886	-.0160	.1060	-.1738	.2154	-.2295
36	-.2898	.4297	-.4892	.4650	-.3679	.2205	-.0517	-.1085	.2342	-.3080	.3031	-.2849	.2053	-.1035	.0001
37	.4620	-.5192	.4442	-.2662	.0383	.1781	-.3301	.3860	-.3416	.2185	-.0566	-.0989	.2098	-.2539	.2296
38	-.5389	.4510	-.2134	-.0785	.3178	-.4252	.3754	-.2010	-.0242	.2159	-.3108	.2874	-.1685	.0087	.1302
39	.5045	-.2458	-.1065	.3802	-.4537	.3108	-.0379	-.2249	.3564	-.3096	.1264	.0918	-.2392	.2572	-.1558
40	-.3659	-.0340	.3820	-.4734	.2754	.0679	-.3382	.3812	-.1948	-.0854	.2825	-.2897	.1266	.0872	-.2185
41	.1518	.3036	-.4978	.3072	.0937	-.3889	.3695	-.0807	-.2366	.3456	-.1898	-.0846	.2614	-.2247	.0318
42	.0936	-.4809	.4054	.0274	-.3979	.3792	-.0242	-.3166	.3403	-.0603	-.2399	.2918	-.0809	-.1710	.2366
43	-.3198	.5121	-.1436	-.3470	.4265	-.0470	-.3445	.3344	.0275	-.3201	.2436	.0774	-.2756	.1609	.1024
44	.4799	-.3877	-.1783	.4764	-.1596	-.3234	.3632	.0485	-.3572	.1952	.1853	-.2937	.0327	.2310	-.1784
45	-.5409	.1456	.4256	-.3447	-.2178	.4208	-.0104	-.3733	.1921	.2378	-.2851	-.0701	.2813	-.0747	-.2036
46	.4902	.1408	-.4950	.0240	.4443	-.1587	-.3531	.2508	.2391	-.2955	-.1213	.2954	.0165	-.2589	.0629
47	-.3383	-.3844	.3574	.3098	-.3631	-.2384	.3566	.1723	-.3391	-.1132	.3124	.0628	-.2784	-.0218	.2394
48	.1166	.5112	-.0735	-.4738	.0303	.4321	.0035	-.3889	-.0307	.3433	.0513	-.2970	-.0653	.2507	.0728
49	.1292	-.4839	-.2459	.3781	.3234	-.2591	-.3600	.1395	.3579	-.0315	-.3238	-.0554	.2670	.1154	-.1980
50	-.3483	.3076	.4595	-.0751	-.4532	-.1348	.3495	.2771	-.1893	-.3300	.0214	.2983	.1120	-.2077	-.1852

TABLE II.- ILLUSTRATION OF SUGGESTED PROCEDURE TO BE USED TO CALCULATE THE TIME RESPONSE TO A UNIT IMPULSE FROM FREQUENCY-RESPONSE DATA

[Values are from fig. 3 and table I at $z = 0.5$ which for a $\Delta\omega = 1$ corresponds to a time of 1.0 sec]

1	2	3	4
n	r_n	$h_n(t)$	2 × 3
1	0.0937	0.8415	0.0788
2	.0549	.0678	.0037
3	.0157	-.7682	-.0121
4	-.0050	-.8979	.0045
5	-.0123	-.2021	.0025
6	-.0135	.6795	-.0092
7	-.0126	.9364	-.0118
8	-.0111	.3324	-.0037
9	-.0096	-.5773	.0055
10	-.0083	-.9562	.0079
11	-.0071	-.4560	.0032
12	-.0061	.4634	-.0028
13	-.0053	.9568	-.0051
14	-.0046	.5705	-.0026
15	-.0042	-.3403	.0014
16	-.0037	-.9382	.0035
17	-.0033	-.6735	.0022
18	-.0029	.2104	-.0006
19	-.0027	.9007	-.0024
20	-.0024	.7631	-.0018
			$\sum$ 4 = 0.0611

$$h(1.0) = \frac{2}{\pi} \Delta\omega \sum 4 = \frac{2}{\pi} \times 1 \times (0.0611) = 0.0390$$

TABLE III.- TABULATION OF THE FUNCTION $\frac{\sin z \sin(2n - 1)z}{z}$ USED TO CALCULATE THE IMAGINARY PART OF THE FOURIER TRANSFORM OF A FUNCTION OF TIME

n	Value of the function $\frac{\sin z \sin(2n - 1)z}{z}$ at values of z of -										
	0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
1	0	0.0500	0.0997	0.1489	0.1974	0.2448	0.2911	0.3359	0.3791	0.4204	0.4597
2	0	.1494	.2950	.4333	.5609	.6746	.7716	.8498	.9074	.9431	.9565
3	0	.2473	.4786	.6791	.8359	.9391	.9826	.9640	.8853	.7521	.5739
4	0	.3428	.6431	.8642	.9789	.9738	.8503	.6248	.3261	-.0081	-.3364
5	0	.4348	.7820	.9721	.9674	.7700	.4210	-.0082	-.4308	-.7622	-.9373
6	0	.5225	.8897	.9932	.8031	.3777	-.1554	-.6374	-.9264	-.9394	-.6765
7	0	.6049	.9619	.9255	.5121	-.1071	-.6775	-.9668	-.8601	-.4058	.2063
8	0	.6814	.9958	.7752	.1402	-.5656	-.9629	-.8415	-.2720	.4350	.8994
9	0	.7510	.9900	.5556	-.2538	-.8857	-.9120	-.3204	.4810	.9466	.7657
10	0	.8131	.9447	.2864	-.6078	-.9889	-.5425	.3514	.9423	.7418	-.0720
11	0	.8671	.8617	-.0084	-.8658	-.8500	.0166	.8579	.8320	-.0244	-.8435
12	0	.9124	.7444	-.3024	-.9871	-.5030	.5698	.9609	.2170	-.7721	-.8395
13	0	.9486	.5975	-.5694	-.9526	-.0329	.9240	.6121	-.5296	-.9355	-.0637
14	0	.9753	.4267	-.7856	-.7676	.4454	.9554	-.0247	-.9550	-.3910	.7708
15	0	.9923	.2388	-.9316	-.4615	.8145	.6531	-.6498	-.8010	.4494	.8965
16	0	.9994	.0415	-.9943	-.0825	.9843	.1226	-.9693	-.1612	.9497	.1980
17	0	.9965	-.1575	-.9683	.3095	.9130	-.4507	-.8330	.5764	.7313	-.6825
18	0	.9836	-.3502	-.8557	.6526	.6182	-.8665	-.3048	.9643	-.0406	-.9355
19	0	.9609	-.5289	-.6668	.8927	.1721	-.9797	.3667	.7674	-.7817	-.3284
20	0	.9286	-.6866	-.4182	.9919	-.3162	-.7506	.8657	.1049	-.9313	.5806
21	0	.8870	-.8169	-.1323	.9345	-.7270	-.2593	.9576	-.6212	-.3761	.9558
22	0	.8366	-.9146	.1654	.7295	-.9599	.3226	.5991	-.9705	.4638	.4523
23	0	.7778	-.9759	.4484	.4094	-.9578	.7918	-.0411	-.7311	.9526	-.4671
24	0	.7112	-.9982	.6913	.0246	-.7211	.9844	-.6621	-.0483	.7205	-.9571
25	0	.6375	-.9808	.8724	-.3640	-.3079	.8331	-.9716	.6639	-.0568	-.5671
26	0	.5575	-.9242	.9757	-.6952	.1807	.3909	-.8242	.9733	-.7912	.3442
27	0	.4718	-.8309	.9917	-.9166	.6250	-.1880	-.2891	.6924	-.9268	.9391
28	0	.3815	-.7044	.9192	-.9933	.9164	-.7012	.3819	-.0086	-.3610	.6706
29	0	.2874	-.5498	.7646	-.9131	.9833	-.9694	.8733	-.7042	.4779	-.2145
30	0	.1903	-.3733	.5417	-.6889	.8095	-.8990	.9539	-.9728	.9552	-.9023
31	0	.0915	-.1819	.2704	-.3559	.4376	-.5145	.5861	-.6513	.7096	-.7606
32	0	-.0084	.0168	-.0251	.0334	-.0416	.0496	-.0576	.0653	-.0730	.0805
33	0	-.1082	.2147	-.3183	.4173	-.5105	.5965	-.6741	.7423	-.8003	.8475
34	0	-.2068	.4041	-.5831	.7354	-.8544	.9350	-.9735	.9690	-.9220	.8349
35	0	-.3034	.5774	-.7957	.9374	-.9892	.9469	-.8152	.6079	-.3459	.0543
36	0	-.3969	.7278	-.9374	.9913	-.8817	.6279	-.2733	-.1220	.4920	-.7763
37	0	-.4866	.8490	-.9953	.8888	-.5584	.0897	.3969	-.7778	.9576	-.8934
38	0	-.5714	.9364	-.9642	.6459	-.0984	-.4798	.8807	-.9618	.6978	-.1898
39	0	-.6503	.9865	-.8471	.3012	.3854	-.8818	.9501	-.5625	-.0902	.6884
40	0	-.7229	.9972	-.6543	-.0913	.7755	-.9757	.5727	.1782	-.8099	.9337
41	0	-.7882	.9683	-.4030	-.4692	.9753	-.7288	-.0740	.8106	-.9170	.3205
42	0	-.8457	.9007	-.1157	-.7731	.9364	-.2273	-.6859	.9512	-.3307	-.5873
43	0	-.8946	.7971	.1819	-.9550	.6682	.3537	-.9753	.5143	.5059	-.9552
44	0	-.9346	.6619	.4633	-.9859	.2364	.8110	-.8059	-.2346	.9596	-.4448
45	0	-.9654	.5001	.7032	-.8614	-.2532	.9851	-.2576	-.8412	.6872	.4745
46	0	-.9864	.3186	.8803	-.6008	-.6809	.8151	.4120	-.9378	-.1054	.9576
47	0	-.9977	.1243	.9789	-.2453	-.9418	.3603	.8877	-.4660	-.8181	.5603
48	0	-.9989	-.0750	.9899	.1488	-.9722	-.2204	.9457	.2884	-.9118	-.3521
49	0	-.9901	-.2712	.9126	.5195	-.7645	-.7240	.5585	.8680	-.3154	-.9408
50	0	-.9715	-.4567	.7537	.8081	-.3696	-.9748	-.0914	.9209	.5196	-.6645

TABLE III.- TABULATION OF THE FUNCTION $\frac{\sin z \sin(2n - 1)z}{z}$.- Concluded

n	Value of the function $\frac{\sin z \sin(2n - 1)z}{z}$ at values of z of -									
	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1	0.4967	0.5314	0.5635	0.5929	0.6195	0.6433	0.6640	0.6818	0.6965	0.7081
2	.9474	.9165	.8650	.7944	.7071	.6057	.4929	.3720	.2462	.1188
3	.3627	.1328	-.1007	-.3228	-.5195	-.6786	-.7911	-.8508	-.8556	-.8069
4	-.6183	-.8202	-.9188	-.9042	-.7806	-.5661	-.2891	.0146	.3071	.5528
5	-.9237	-.7273	-.3909	.0155	.4090	.7117	.8656	.8442	.6571	.3468
6	-.2196	.2932	.7097	.9094	.8385	.5245	.0661	-.3982	-.7319	-.8415
7	.7244	.9397	.7706	.2937	-.2904	-.7423	-.8826	-.6632	-.1838	.3535
8	.8768	.3878	-.2975	-.8096	-.8796	-.4812	.1614	.6996	.8508	.5472
9	.0710	-.6586	-.9297	-.5689	.1659	.7704	.8410	.3453	-.3662	-.8090
10	-.8124	-.8652	-.1999	.6162	.9031	.4362	-.3781	-.8565	-.6140	.1261
11	-.8080	.0316	.8228	.7784	-.0382	-.7959	-.7436	.0439	.7632	.7040
12	.0794	.8881	.6401	-.3516	-.9085	-.3897	.5697	.8366	.1205	-.7121
13	.8800	.6120	-.4803	-.8979	-.0904	.8186	.5968	-.4240	-.8411	-.1114
14	.7190	-.4446	-.8971	.0464	.8957	.3419	-.7235	-.6438	.4234	.8048
15	-.2278	-.9342	.0004	.9137	.2171	-.8386	-.4104	.7166	.5674	-.5584
16	-.9256	-.2325	.8973	.2642	-.8650	-.2929	.8292	.3183	-.7902	-.3393
17	-.6119	.7657	.4797	-.8239	-.3395	.8557	.1967	-.8612	-.0556	.8414
18	.3704	.7874	-.6407	-.5443	.8169	.2430	-.8799	.0739	.8265	-.3610
19	.9480	-.1942	-.8224	.6388	.4551	-.8699	.0309	.8278	-.4788	-.5416
20	.4896	-.9288	.2006	.7615	-.7526	-.1922	.8720	-.4500	-.5176	.8110
21	-.5038	-.4780	.9297	-.3800	-.5608	.8811	-.2556	-.6239	.8128	-.1334
22	-.9467	.5823	.2968	-.8904	.6737	.1399	-.8065	.7328	-.0079	-.7001
23	-.3550	.9001	-.7710	.0772	.6561	-.8894	.4626	.2909	-.8077	.7160
24	.6246	.0700	-.7092	.9170	-.5809	-.0888	.6873	-.8650	.5301	.1041
25	.9217	-.8494	.3915	.2336	-.7388	.8945	-.6397	.1021	.4649	-.8026
26	.2114	-.6855	.9186	-.8375	.4757	.0366	-.5225	.8186	-.8307	.5639
27	-.7297	.3525	.0991	-.5190	.8062	-.8966	.7743	-.4741	.0722	.3332
28	-.8735	.9410	-.8656	.6607	-.3616	.0158	.3229	-.6032	.7840	-.8413
29	-.0627	.3285	-.5628	.7436	-.8574	.8957	-.8576	.7482	-.5791	.3669
30	.8167	-.7029	.5641	-.4078	.2403	-.0681	-.1019	.2632	-.4096	.5359
31	.8030	-.8383	.8647	-.8823	.8914	-.8918	.8838	-.8678	.8440	-.8130
32	-.0887	.0947	-.1015	.1080	-.1142	.1202	-.1258	.1311	-.1361	.1407
33	-.8834	.9070	-.9190	.9190	-.9075	.8847	-.8514	.8082	-.7559	.6958
34	-.7135	.5626	-.3902	.2044	-.0142	-.1718	.3452	-.4983	.6249	-.7198
35	.2359	-.4993	.7102	-.8495	.9055	-.8747	.7625	-.5818	.3519	-.0967
36	.9274	-.9244	.7701	-.4932	.1423	.2229	-.5417	.7626	-.8523	.8004
37	.6054	-.1707	-.2982	.6819	-.8854	.8616	-.6229	.2352	.1992	-.5694
38	-.3781	.8007	-.9297	.7250	-.2676	-.2732	.7022	-.8695	.7236	-.3264
39	-.9484	.7510	-.1992	-.4354	.8475	-.8458	.4420	.1599	-.6671	.8411
40	-.4824	-.2564	.8231	-.8730	.3875	.3226	-.8160	.7969	-.2923	-.3735
41	.5109	-.9369	.6396	.1386	-.7927	.8268	-.2317	-.5221	.8560	-.5301
42	.9458	-.4226	-.4809	.9201	-.4996	-.3708	.8758	-.5597	-.2612	.8148
43	.3471	.6306	-.8968	.1742	.7220	-.8052	.0068	.7763	-.6872	-.1480
44	-.6309	.8796	.0011	-.8609	.6018	.4179	-.8773	.2069	.7055	-.6916
45	-.9195	.0068	.8975	-.4669	-.6369	.7808	.2201	-.8703	.2310	.7237
46	-.2033	-.8747	.4791	.7022	-.6919	-.4635	.8206	.1886	-.8548	.0894
47	.7351	-.6407	-.6412	.7055	.5390	-.7538	-.4315	.7847	.3217	-.7981
48	.8702	.4103	-.8221	-.4624	.7681	.5075	-.7094	-.5451	.6469	.5748
49	.0544	.9381	.2014	-.8627	-.4303	.7242	.6143	-.5369	-.7399	.3196
50	-.8209	.2694	.9298	.1691	-.8290	-.5498	.5511	.7891	-.1684	-.8408

TABLE IV.- ILLUSTRATION OF SUGGESTED PROCEDURE TO BE USED TO CALCULATE THE FOURIER TRANSFORM OF A FUNCTION OF TIME

[Values are from fig. 6, table I, and table III at $z = 0.25$ which for a $\Delta t = 0.1$ corresponds to a frequency of 5.0 radians/sec]

①	②	③	④	⑤	⑥
n	r_n	$R[F(i\omega)]$	$I[F(i\omega)]$	② × ③	② × ④
1	0.0420	0.9588	0.2448	0.0403	0.0103
2	.0950	.7241	.6746	.0688	.0641
3	.1165	.3120	.9391	.0364	.1094
4	.1200	-.1764	.9738	-.0212	.1169
5	.1120	-.6216	.7700	-.0696	.0862
6	.1000	-.9147	.3777	-.0915	.0378
7	.0860	-.9838	-.1071	-.0846	-.0092
8	.0720	-.8120	-.5656	-.0585	-.0407
9	.0580	-.4415	-.8857	-.0256	-.0514
10	.0460	.0372	-.9889	.0017	-.0455
11	.0360	.5068	-.8500	.0182	-.0306
12	.0290	.8522	-.5030	.0247	-.0146
13	.0220	.9891	-.0329	.0218	-.0007
14	.0170	.8837	.4454	.0150	.0076
15	.0130	.5620	.8145	.0073	.0106
16	.0100	.1027	.9843	.0010	.0098
17	.0065	-.3817	.9130	-.0025	.0059
18	.0050	-.7727	.6182	-.0039	.0031
19	.0040	-.9745	.1721	-.0039	.0007
20	.0030	-.9377	-.3162	-.0028	-.0010
21	.0020	-.6714	-.7270	-.0013	-.0015
22	.0015	-.2406	-.9599	-.0004	-.0014
23	.0010	.2491	-.9578	.0003	-.0010
24	.0005	.6777	-.7211	.0003	-.0004
25	.0002	.9405	-.3079	.0002	-.0001
				$\sum$ ⑤ = -0.1297	$\sum$ ⑥ = 0.2644

$$R[F(15.0)] = \Delta t \sum ⑤ = 0.1 \times (-0.1297) = -0.01297 \quad \text{(See eq. (19).)}$$

$$I[F(15.0)] = -\Delta t \sum ⑥ = -0.1 \times (0.2644) = -0.02644 \quad \text{(See eq. (20).)}$$

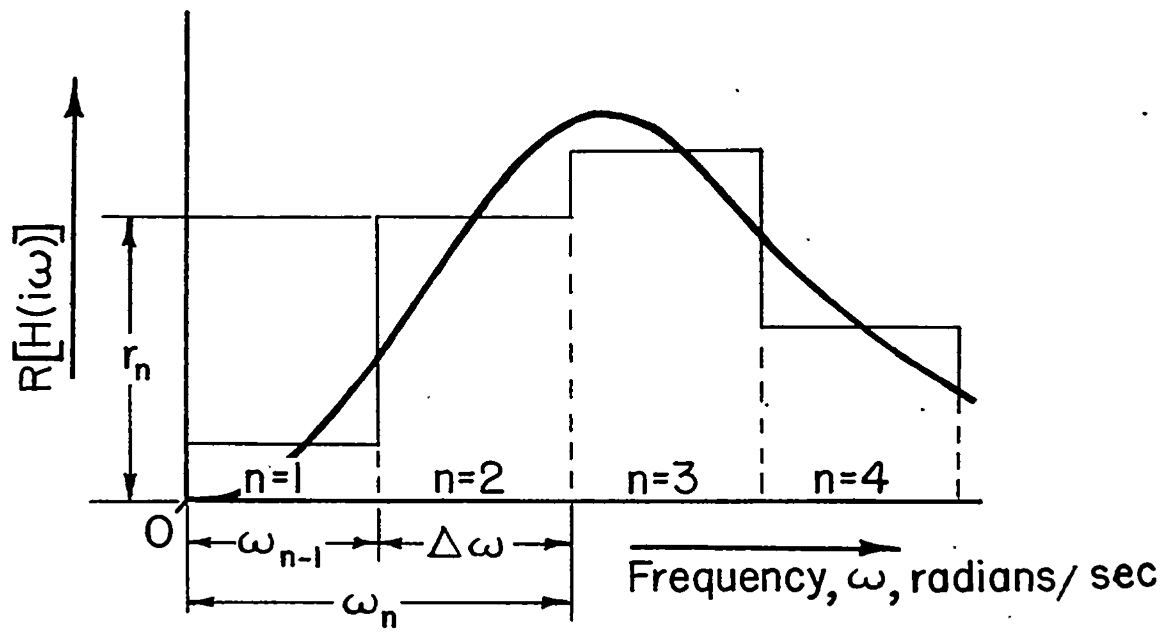
$$A(5.0) = \sqrt{R[F(i\omega)]^2 + I[F(i\omega)]^2} = 0.02945 \quad \text{(See eq. (21).)}$$

$$\Phi(5.0) = -\tan^{-1} \frac{I[F(i\omega)]}{R[F(i\omega)]} = -\tan^{-1} 2.0387 \quad \text{or} \quad \Phi = -116^\circ 08' \quad \text{(See eq. (22).)}$$

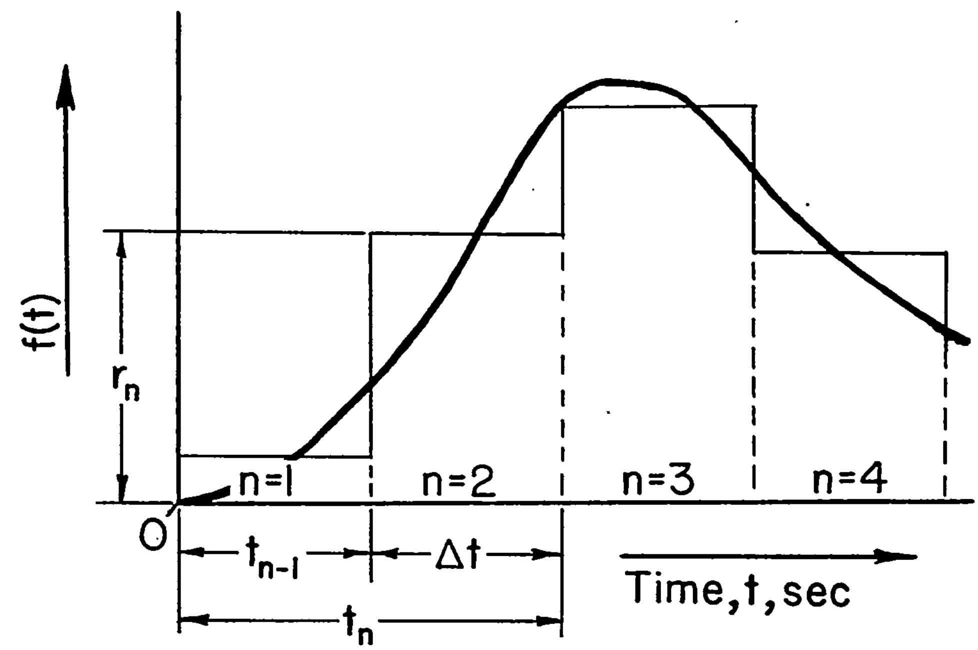
TABLE V.- INTEGRATING MATRIX $[\Omega_{ij}]$

$[\Omega_{ij}] = \Delta t$

0.000000																							
.500000	0.500000																						
.333333	1.333333	0.333333																					
.375000	1.125000	1.125000	0.375000																				
.311111	1.422222	.533333	1.422222	0.311111																			
.329861	1.302083	.868056	.868056	1.302083	0.329861																		
.311111	1.422222	.533333	1.422222	.644444	1.333333	0.333333																	
.329861	1.302083	.868056	.868056	1.302083	.663194	1.333333	0.333333																
					.704861	1.125000	1.125000	0.375000															
					.640972	1.422222	.533333	1.422222	0.311111														
					.659722	1.302083	.868056	.868056	1.302083	0.329861													
					.704861	1.125000	1.125000	.750000	1.125000	1.125000	0.375000												
					.659722	1.302083	.868056	.868056	1.302083	.663194	1.333333	0.333333											
										.704861	1.125000	1.125000	0.375000										
										.640972	1.422222	.533333	1.422222	0.311111									
										.659722	1.302083	.868056	.868056	1.302083	0.329861								
										.704861	1.125000	1.125000	.750000	1.125000	1.125000	0.375000							
										.659722	1.302083	.868056	.868056	1.302083	.663194	1.333333	0.333333						
															.704861	1.125000	1.125000	0.375000					
															.640972	1.422222	.533333	1.422222	0.311111				
															.659722	1.302083	.868056	.868056	1.302083	0.329861			
															.704861	1.125000	1.125000	.750000	1.125000	1.125000	0.375000		
															.659722	1.302083	.868056	.868056	1.302083	.663194	1.333333	0.333333	. . .
																				.704861	1.125000	1.125000	. . .
																				.640972	1.422222	.533333	. . .
																				.659722	1.302083	.868056	. . .



(a) Terms used to obtain the time response to a unit impulse from frequency-response data.



(b) Terms used to obtain the Fourier transform of a function of time.

Figure 1.- Illustration of terms used in the development of the methods of this paper.

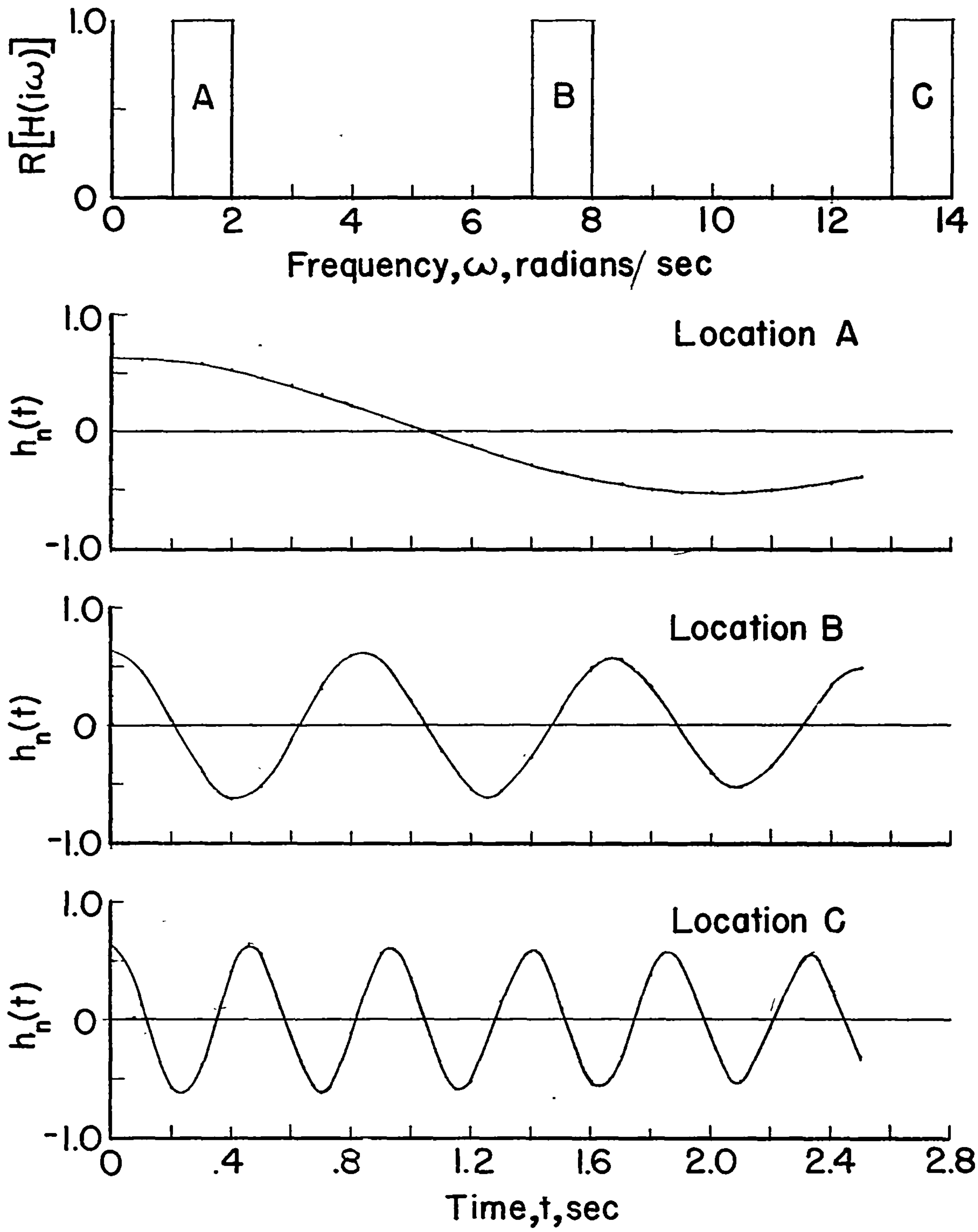


Figure 2.- Time response to a unit impulse for an element of the staircase function at various locations in the frequency plane.

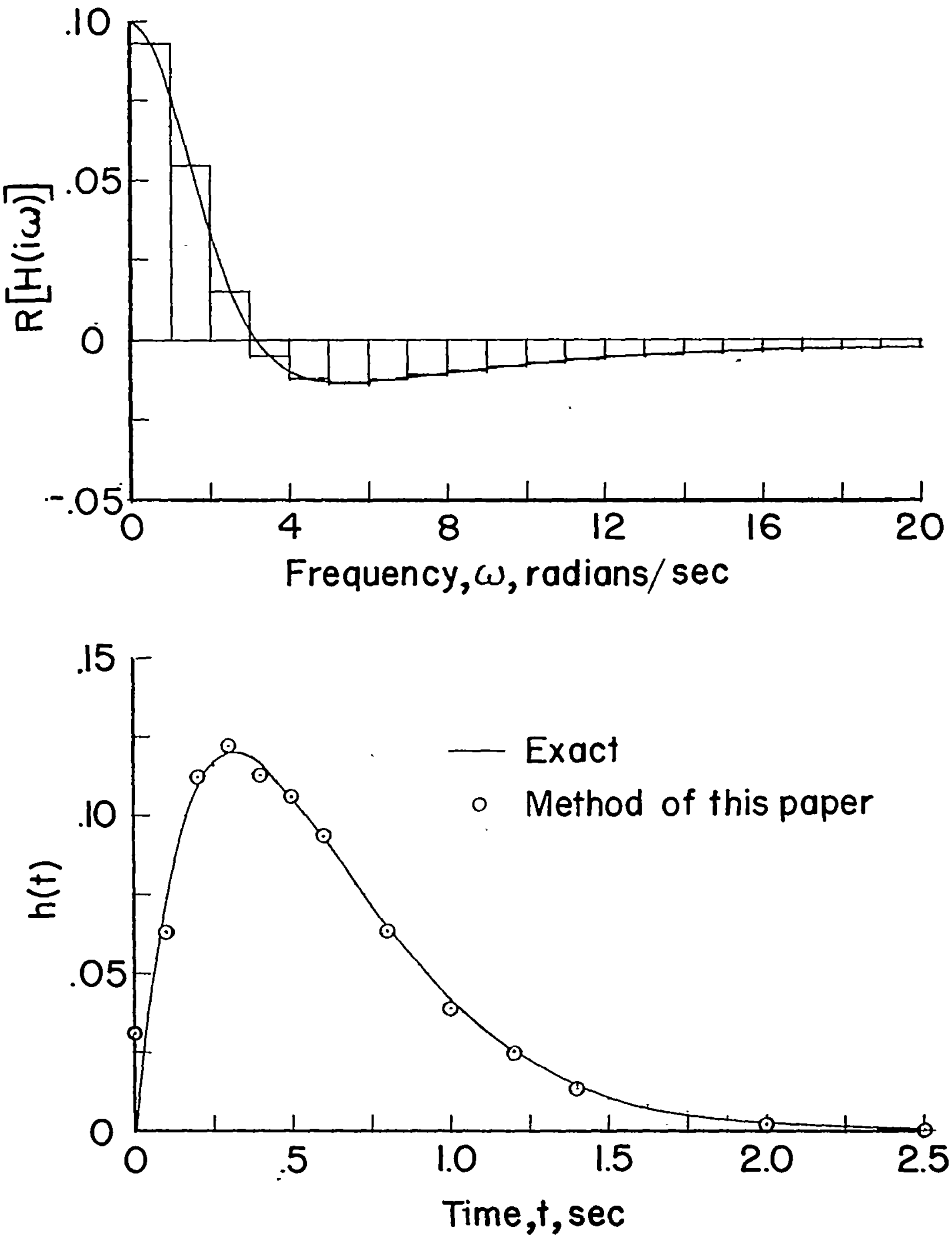


Figure 3.- Comparison of the response to a unit impulse calculated by the method of this paper with the exact response for a simple system the transfer function of which is $\frac{x}{\delta} = \frac{1}{s^2 + 6s + 10}$.

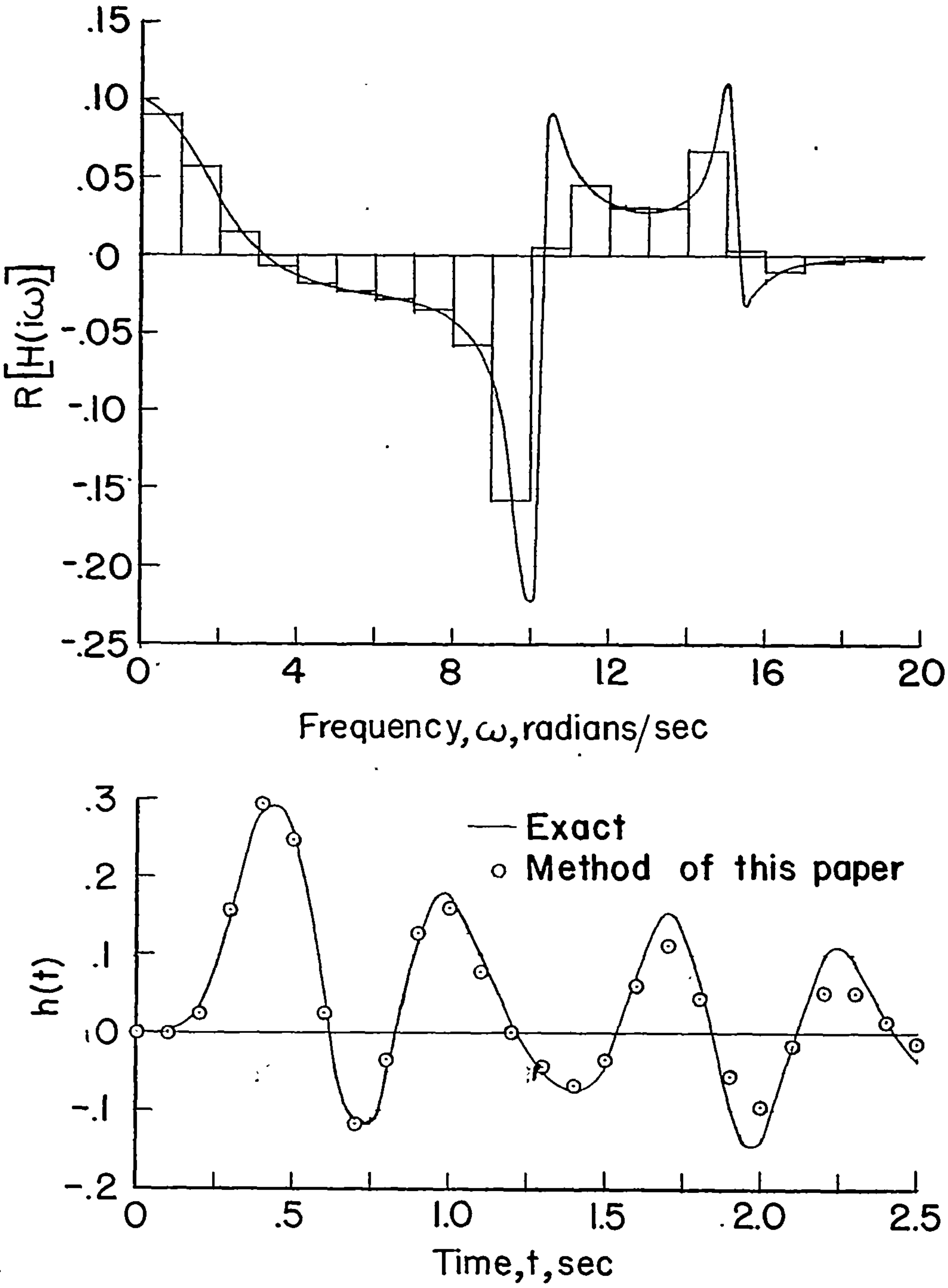


Figure 4.- Comparison of the response to a unit impulse calculated by the method of this paper with the exact response for a multiple-mode system the transfer function of which is

$$\frac{x}{\delta} = \frac{1}{s^2 + 6s + 10} \, \frac{100}{s^2 + 0.4s + 100} \, \frac{225}{s^2 + 0.2s + 225}.$$

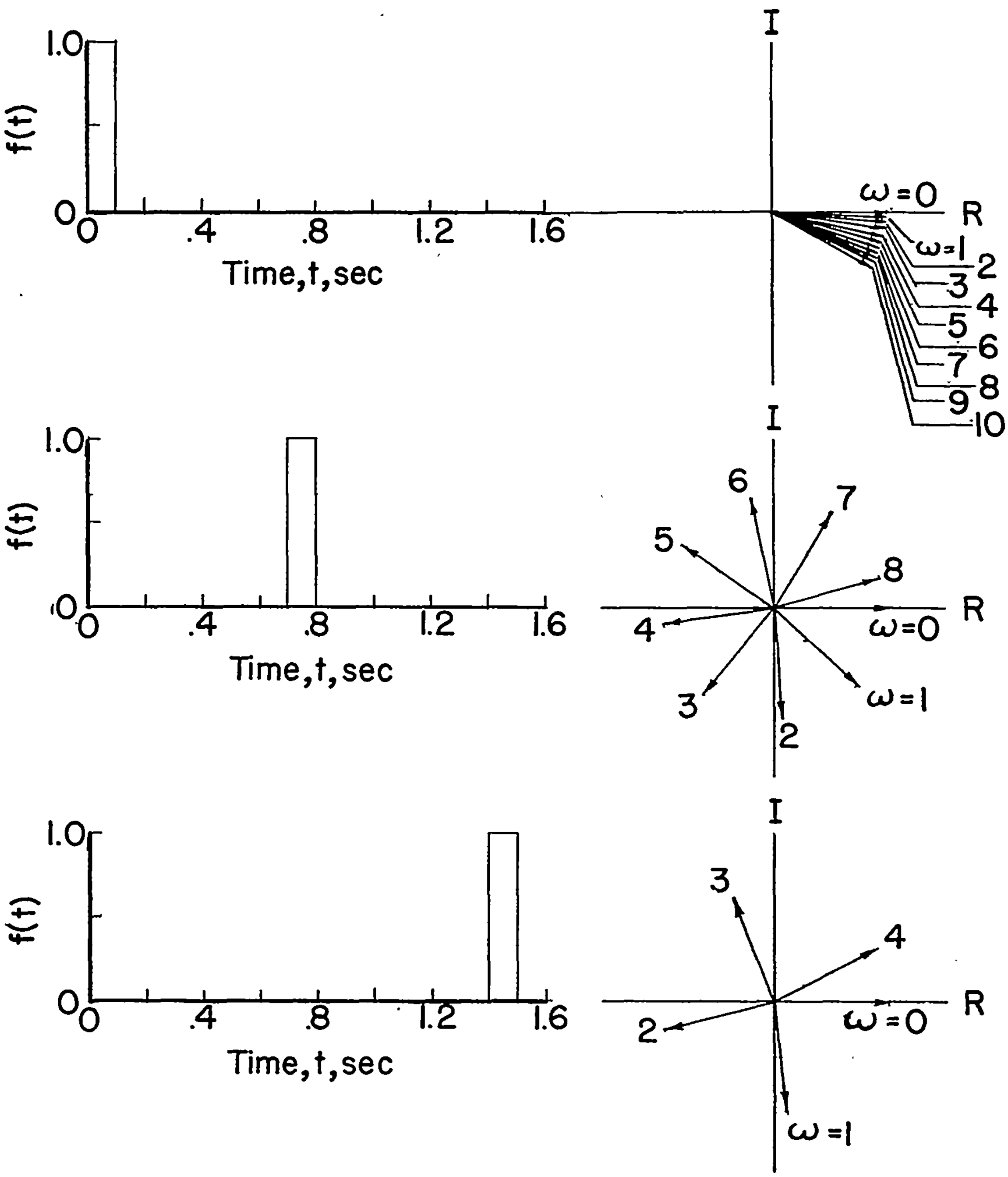


Figure 5.- Vector representation of the frequency content of an element of the staircase function at various locations in the time plane.

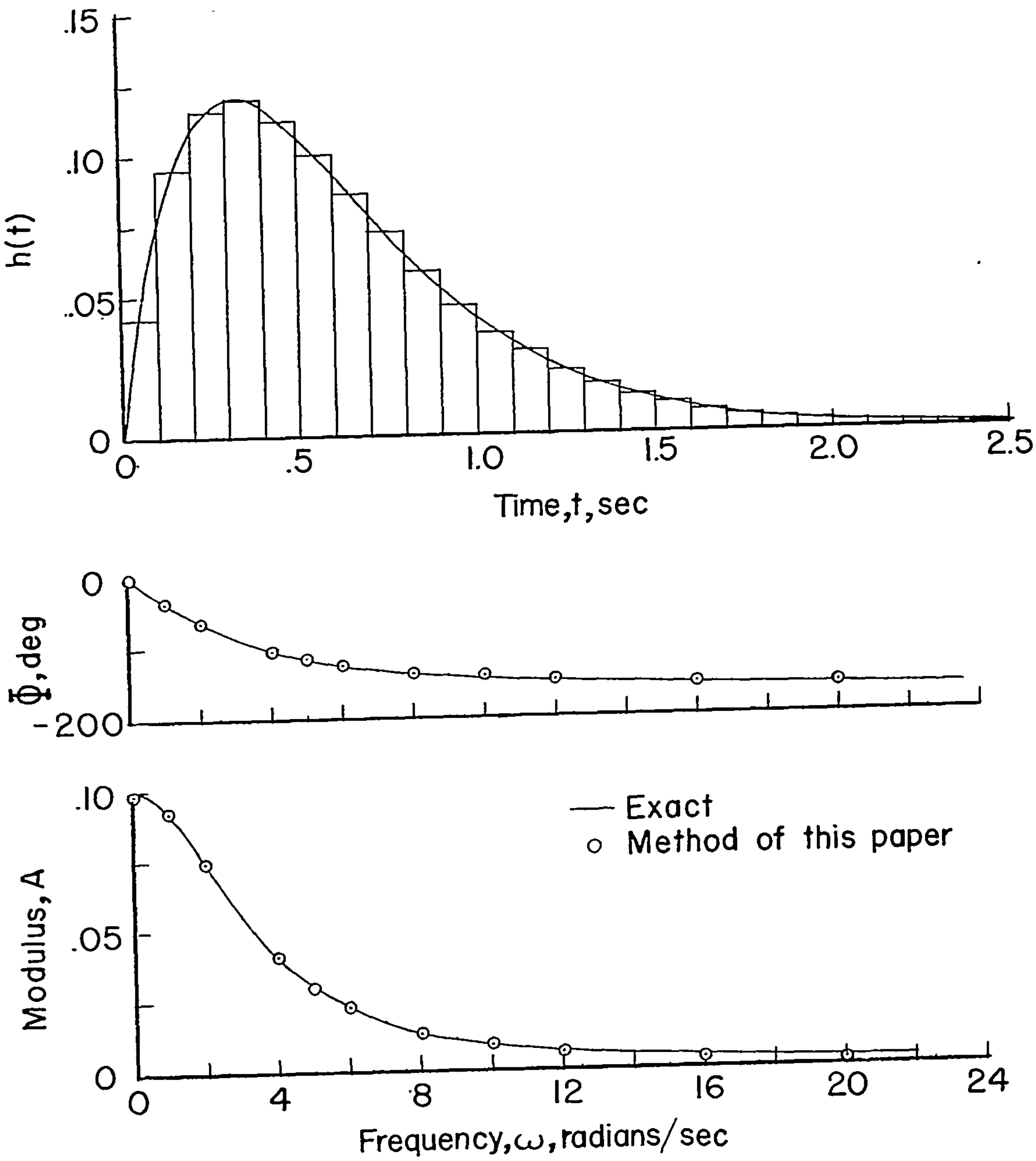


Figure 6.- Comparison of the modulus and phase angle of the Fourier transform of a time function calculated by the method of this paper with the exact values.

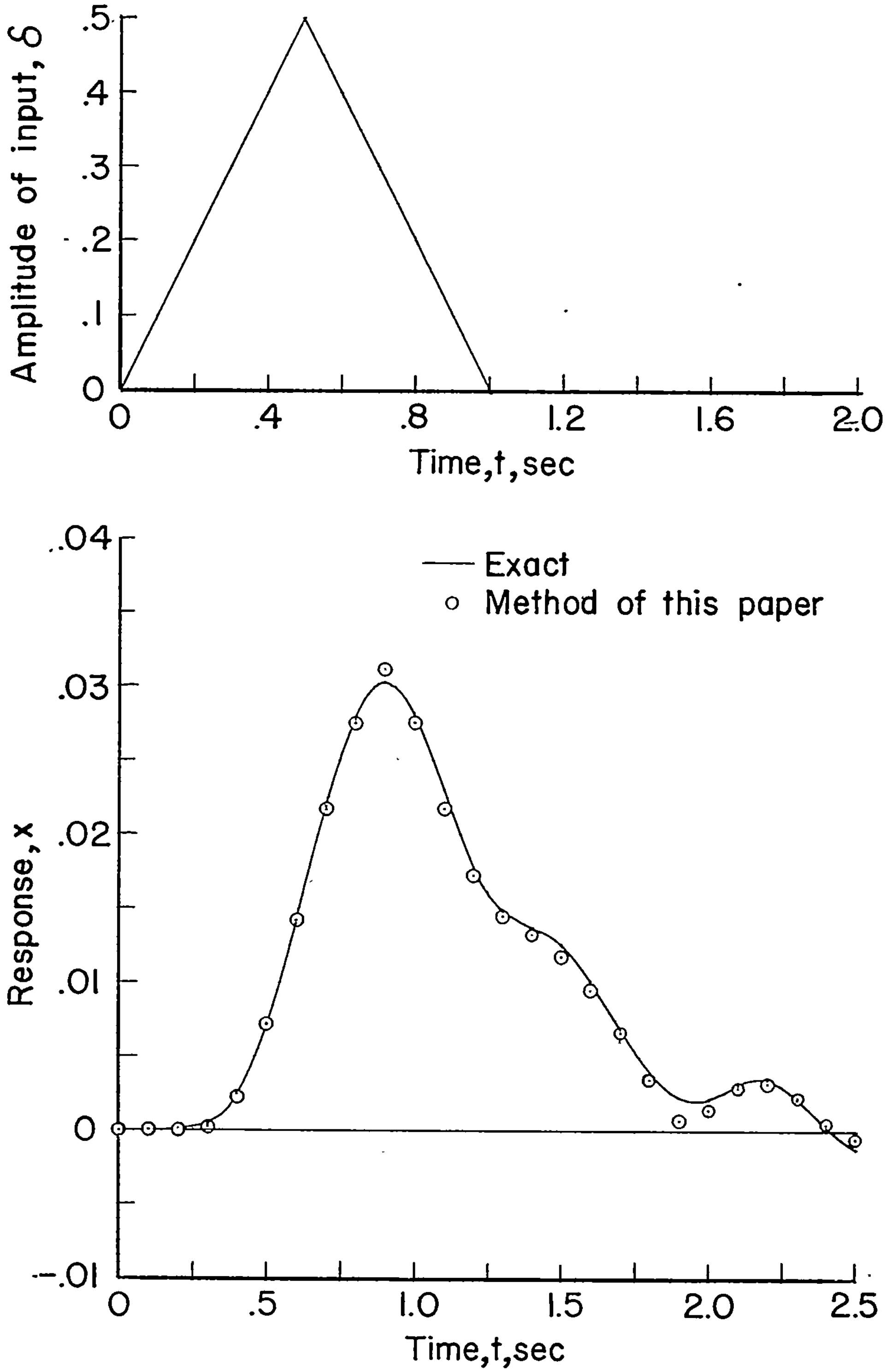
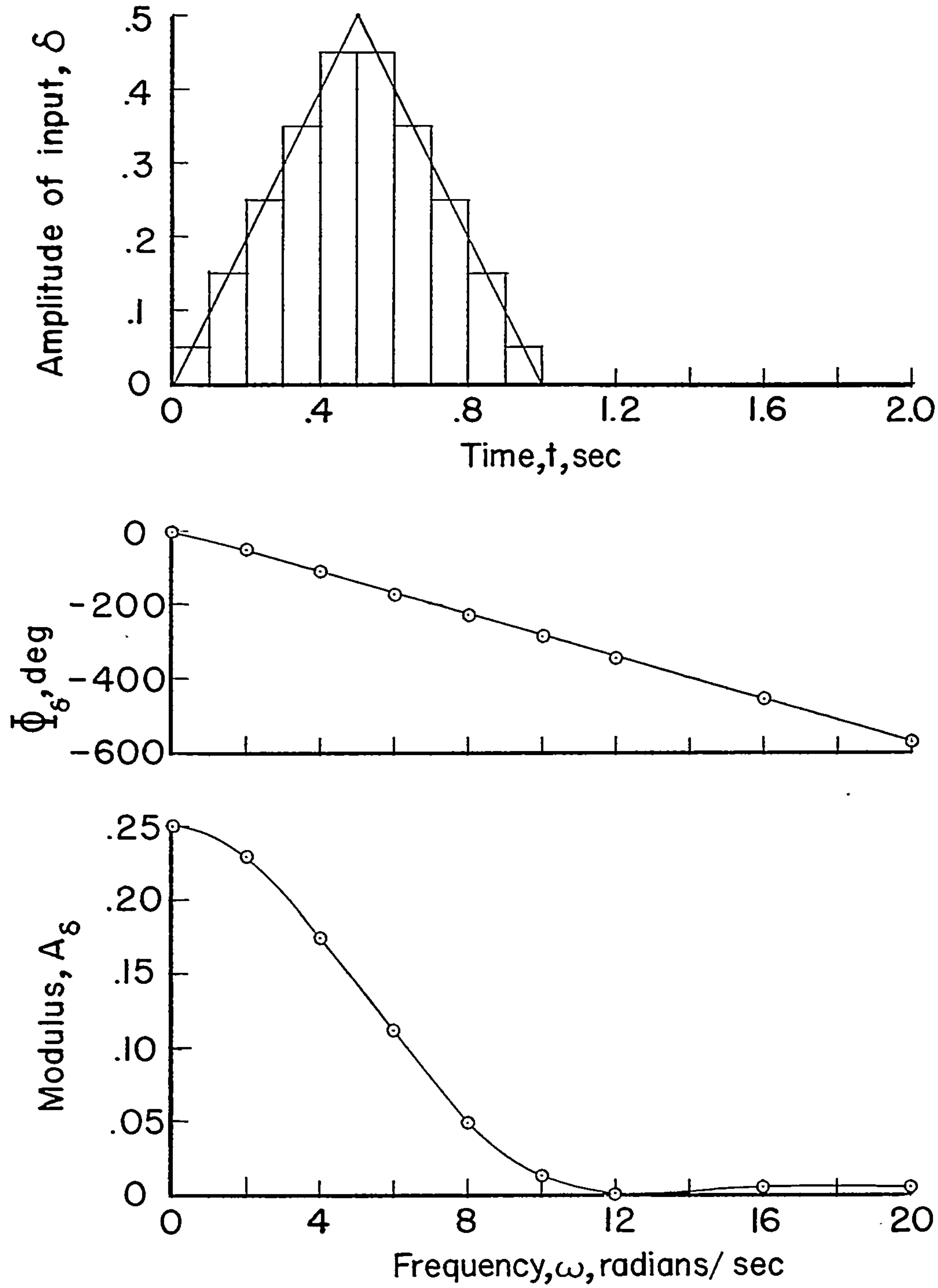
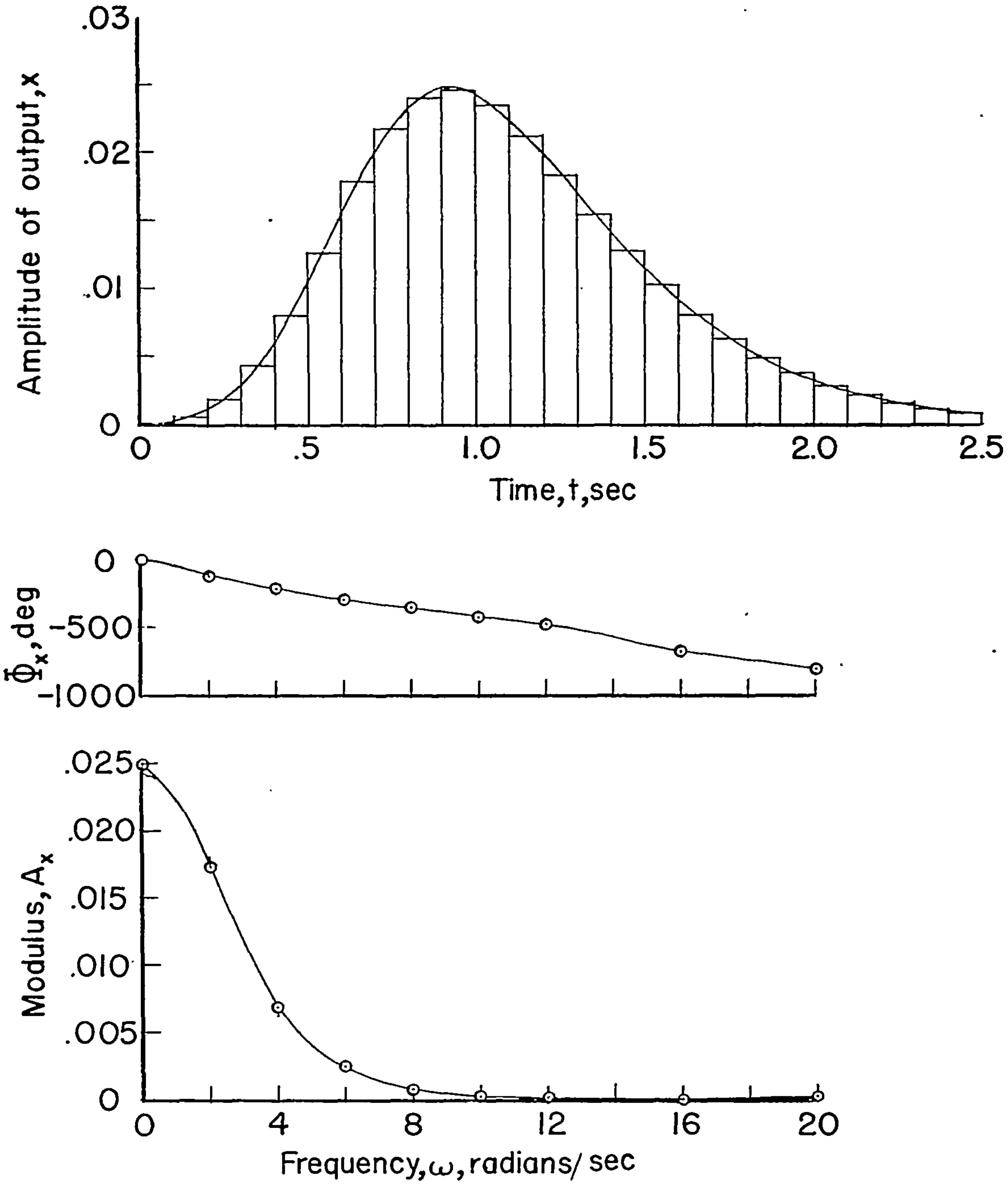


Figure 7.- Comparison of the response to an arbitrary input calculated by using the $h(t)$ obtained by the method of this paper with that calculated by using the exact $h(t)$ for the multiple-mode system.



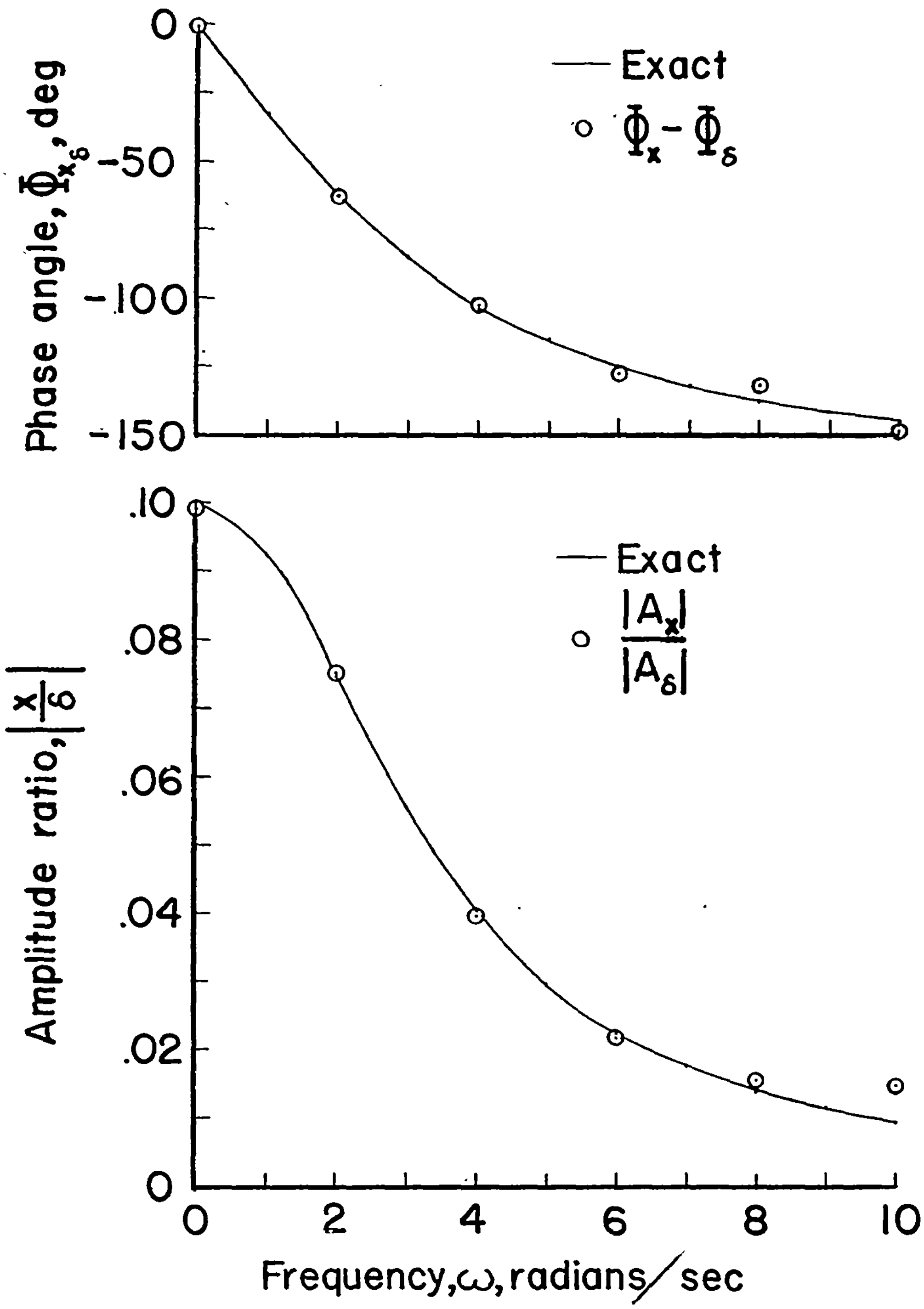
(a) Modulus and phase angle of input.

Figure 8.- Comparison with the exact values of the frequency response obtained by application of the method of this paper to an input and output.



(b) Modulus and phase angle of output.

Figure 8.- Continued.



(c) Amplitude ratio and phase angle of frequency response.

Figure 8.- Concluded.

NACA - Langley Field, Va.